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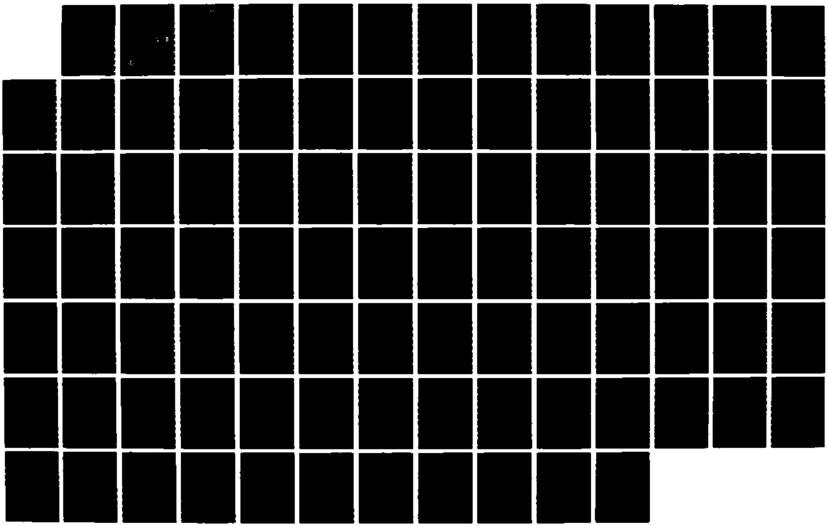

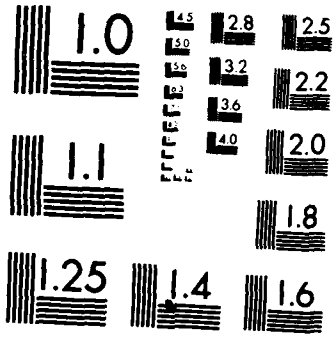

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NSWC TR 84-342

COMPARISONS OF FINITE ELEMENT AND BENCHMARK ELASTIC SOLUTIONS FOR CYLINDRICAL SHELLS UNDER CONCENTRATED LOADS

AD-A170 564

BY MINOS MOUSSOUROS

RESEARCH AND TECHNOLOGY DEPARTMENT

JANUARY 1985

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This report summarizes linear solutions to the pinched circular cylinder problem as provided by the "technical" or "engineering theories of shells. Comparisons of these analytical solutions with finite element analyses by ABAQUS and STAGS with reasonably sized finite element grids are tabulated. The agreement between the analytical and the finite element results is quite good.

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FOREWORD

The purpose of this study is to survey the analytical linear solutions for a thin cylindrical shell loaded by point forces and compare them to various finite element models. The shell theories used to quantify maximum central displacement are the so-called "technical" or "engineering" theories. A computer program was developed to automate the procedure, and a variety of plots for various radius-to-thickness and length-to-radius ratios was obtained. The finite element solutions were computed using general purpose computer programs STAGS and ABAQUS. The agreement between the analytical and the finite element results is quite good.

This work was sponsored by NAVSEA (63R) through the Undersea Weaponry Warheads, and Fuzes Block Program.

Approved by:

KURT F. MUELLER, Head
Energetic Materials Division

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CHAPTER 1

INTRODUCTION

The validation of finite element computer programs for various configurations, loads, and material properties is of great interest to the Navy, which is applying these methods on a regular basis.

Structural statics problems, depending on the loading, material properties, and geometry changes during deformation can be classified as geometrically linear or nonlinear, elastic, plastic or elastoplastic with small or large deflections and/or rotations. For convenience we will refer to three ranges of interest, i.e., (1) the small deflection elastic material regime, (2) the large deflection and/or rotations, fully plastic material regime, and (3) an intermediate range with elastoplastic properties and deformations ranging from small to large. It is clear that all three ranges must be validated.

This report addresses two specific structural problems, and focuses on the first and simplest range of interest, i.e., the small-deflection linear regime. The two problems are:

1. A circular cylindrical shell simply supported at both its ends and subjected to a centrally located inward concentrated static load.

2. A self-equilibrating circular cylindrical shell subjected to two inward pinching static loads at equal and opposite radial positions.

The report also presents a comparison of finite element solutions obtained using both the STAGS[1] and ABAQUS[2] computer codes with analytical solutions found in the literature for thin shells.[3,4,5,7,8-12] The analytical methods are derived from the so-called "engineering" or "technical" theories of thin shells, which differ in the complexity and type of results that can be generated from solutions based on three-dimensional elasticity theory methods. The two loading cases (1) and (2) are illustrated in Figures 1 and 2 and labeled as the "point load case" and the "pinched load case," respectively. Other loading and boundary conditions are examined in an extensive review of the pertinent literature (References 3 to 80), some of which is included as Appendix A. Finally a variety of plots for various radius-to-thickness and length-to-radius ratios were obtained. They can be used for design purposes without recourse to a computer, since they are given for nondimensional variables and a range of interest to marine structures.

Future studies will address the same two problems for the second range of interest, i.e., for a fully plastic shell treated as if made of a "rigid" plastic material.

CHAPTER 2

GOVERNING EQUATIONS

The analytical solutions with which the numerical results are later compared, follow the works of Odqvist,[30] Bijlaard,[36] Mizoguchi,[38] Yuan-Ting,[25] and Timoshenko.[4] These authors (except Timoshenko in this case) developed partial differential equations governing the response of the cylinder under the loads of interest, that represent simplifications of the more general detailed governing equations as represented in the works of Fluegge.[3] A detailed account of the development of some of these equations employed in this report is presented in Appendix B.

Table 1 summarizes the derived partial differential equations used in this study, and on which the computed theoretical solutions are based. Table 2 is an additional summary and supplements Table 1. Timoshenko's solution was developed by employing the principle of virtual work and hence no differential equation was directly solved and given in Table 1. Further details on all the solutions can be found in Appendix B and the references cited therein.

CHAPTER 3

ANALYTICAL SOLUTIONS

This portion of the report deals with the employed analytical solutions. In Odqvist's, Bijlaard's, Mizoguchi's, and Yuan-Ting's case we start from the equations of equilibrium for an infinitesimal shell element, introduce the strain-displacement equations, and follow the approach outlined in Appendix B to reduce the equations, to a partial differential equation in one variable after neglecting certain terms.

Here it is sufficient to say that, depending on the type of assumptions made with respect to the thickness-to-radius ratio (h/R), the resulting equations can be simplified accordingly. Consequently, we can have on the one hand the full Fluegge[3] equations, which are the most complicated, while on the other hand the Donnell[34] equations, which are the most simplified consistent version. We also note that the even simpler equation given by Schorer[37] and Odqvist[30] will be used for numerical comparisons later on. In addition there are other variations with respect to complexity. Further discussion can be found in Appendices A, B, and C and the references therein.

Figure 1 displays the point load case. The cylinder is loaded at midlength (L/2) by an inwards concentrated force P and is simply supported at its two extreme ends x = 0 and x = L.

Figure 2 displays the pinched load case. The cylinder is loaded at midlength (L/2) by two equal and opposite inwards directed forces P at the angular locations $\varphi = 0$ and $\varphi = \pi$. The ends x = 0 and x = L are free to deform.

This report uses the following four solutions (see Appendix B and Table 2) for the central deflection of a simply supported circular cylindrical shell of radius R, length L, skin thickness h, ana material with Young's Modulus E and poisson's ratio ν subject to a point load P (Figure 1). The deflection (w) is computed under the load.

ODQVIST SOLUTION[30]

$$W = \left\{ \frac{\sqrt{2-\sqrt{2}}\,[12(1-\nu^2)]^{5/8}\, R^{3/4}\, L^{1/2}}{2\pi\sqrt{2\pi}\, h^{9/4}} \right\} \left(\frac{P}{E}\right) \sum_{n=1,3,5}^{\infty} \frac{1}{n\sqrt{n}} \qquad (1)$$

where the infinite sum can be calculated as 1.68876336 correct to six decimal places.

BIJLAARD'S SOLUTION[36]

$$W = \sum_{m=0,1}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{(m^2\alpha^2 + n^2\pi^2)^2}{DG} L^4 Z_{m,n} \tag{2}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{3}$$

$$Z_{m,n} = (-1)^{\frac{n-1}{2}} \frac{P}{\pi RL} \qquad \text{for } m = 0 \qquad \text{and} \qquad n = 1,3,5,7,\ldots, \tag{4}$$

$$Z_{m,n} = (-1)^{\frac{n-1}{2}} \frac{2P}{\pi RL} \qquad \text{for } m = 1,2,3\ldots, \quad \text{and} \quad n = 1,3,5,\ldots, \tag{5}$$

and

$$G = (m^2\alpha^2 + n^2\pi^2)^4 + 12(1-\nu^2)n^4\pi^4\alpha^4\gamma^2 - m^2\alpha^4 \left\{2m^4a^4 + (6+\nu-\nu^2)n^4\pi^4 + (7+\nu)m^2\alpha^2n^2\pi^2\right\} \tag{6}$$

MIZOGUCHI'S SOLUTION[38]

$$W = \frac{R^3P}{\pi DL} \left\{ \sum_{m=1,3,5}^{\infty} \left[\frac{1}{\mu^4 + 12(1-\nu^2)\left(\frac{R}{h}\right)^2} \right] + 2 \sum_{m=1,3,5,}^{\infty} \sum_{n=1,2,3}^{\infty} \frac{(A\mu^4 + 2U\mu^2n^2 + Bn^4)}{H} \right\} \tag{7}$$

where

$$\mu = m\pi a/L \tag{8}$$

$$A = 1 + \frac{1}{3}\left(\frac{h}{R}\right)^2 \quad , \; B = 1 + \frac{1}{12}\left(\frac{h}{R}\right)^2 \quad , \; C = 1 + \frac{1-\nu^2}{72}\left(\frac{h}{R}\right)^2$$

$$F = 2(1-\nu^2)\left(\frac{R}{h}\right)^2 A \quad , \qquad U = 1 + \left\{\frac{1 + (1-\nu)^2}{12(1-\nu)}\right\}\left(\frac{h}{R}\right)^2 \tag{9}$$

and

$$H = A\mu^8 + 4B\mu^6 n^2 + 6\,C\mu^4 n^4 - (8 - 2\nu^2)\,\mu^4 n^2 + 6F\mu^4 + 4\mu^2\,(n^2-1)^2 n^2 + n^4\,(n^2 - 1)^2 \tag{10}$$

YUAN-TING'S SOLUTION[25]

$$W = \frac{6(1-\nu^2)}{\pi}\left(\frac{R}{h}\right)^2 \frac{P}{Eh} \sum_{n=2,4,6}^{\infty} \left[\left\{ \frac{\cosh\left(B_1 \frac{L}{R}\right) - \cos\left(A_1 \frac{L}{R}\right)}{\sinh^2\left(B_1 \frac{L}{R}\right) + \sin^2\left(A_1 \frac{L}{R}\right)} \right\} \left\{ M_1 \sinh\left(B_1 \frac{L}{R}\right) - M_2 \sin\left(A_1 \frac{L}{R}\right) \right\} + \left\{ \frac{\cosh\left(G_1 \frac{L}{R}\right) - \cos\left(C_1 \frac{L}{R}\right)}{\sinh^2\left(G_1 \frac{L}{R}\right) + \sin^2\left(C_1 \frac{L}{R}\right)} \right\} \left\{ M_3 \sinh\left(G_1 \frac{L}{R}\right) - M_4 \sin\left(C_1 \frac{L}{R}\right) \right\} \right] \tag{11}$$

where

$$M_1 = \frac{K_1}{A_1 B_1 (A_1^2 + B_1^2) H}, \quad M_2 = \frac{K_2}{A_1 B_1 (A_1^2 + B_1^2) H}$$

$$M_3 = \frac{K_3}{C_1 G_1 (C_1^2 + G_1^2) H}, \quad M_4 = \frac{K_4}{C_1 G_1 (C_1^2 + G_1^2) H}$$

$$H = (n_1^2 + n_2^2)(n_1^2 + n_3^2)$$

$$K_1 = \Phi_1(n_4^2 - 4A_1^2B_1^2) + 4\Phi_2 n_4 A_1 B_1$$

$$\Phi_1 = A_1(n_1^2 - n_2 n_3) - B_1 n_1(n_2 + n_3)$$

$$K_2 = \Phi_2(n_4^2 - 4A_1^2B_1^2) - 4\Phi_1 n_4 A_1 B_1$$

$$\Phi_2 = B_1(n_1^2 - n_2 n_3) + A_1 n_1(n_2 + n_3)$$

$$K_3 = \Phi_3(n_3^2 - 4C_1^2G_1^2) + 4\Phi_4 n_5 C_1 G_1$$

$$\Phi_3 = C_1(n_1^2 + n_2 n_3) + G_1 n_1(n_2 - n_3)$$

$$K_4 = \Phi_4(n_3^2 - 4C_1^2G_1^2) - 4\Phi_3 n_5 C_1 G_1$$

$$\Phi_4 = G_1(n_1^2 + n_2 n_3) - C_1 n_1(n_2 - n_3)$$

$$n_1 = A_1^2 - B_1^2 - C_1^2 + G_1^2$$

$$n_2 = 2(A_1B_1 + C_1G_1)$$

$$n_3 = 2(A_1B_1 + C_1G_1)$$

$$n_4 = A_1^2 - B_1^2 + n^2$$

$$n_5 = C_1^2 - G_1^2 + n^2 \qquad (12)$$

and

A_1, C_1 = real parts of complex roots

B_1, G_1 = imaginary parts of complex roots of the characteristic equation.

Furthermore, the inextensional solution[30] for the pinched load case is:

$$w = \frac{3(1-\nu^2)}{2}\left(\frac{\pi^2-8}{\pi}\right)\left(\frac{R}{h}\right)^3 \frac{P}{EL} \tag{13}$$

This solution was obtained by enforcing the inextensibility condition to the three strain components at the shell's middle surface. The assumed tangential displacement does not satisfy the boundary conditions at the free ends. We note that for small values of L, Equation (13) is of the same form as the "ring formula" of Reference 58 (p. 220). However, from physical grounds the deflection under the load should not become zero as L increases, but have a finite value, which means that Equation (13) is not valid for large L.

To complete this section, we note that the case of an infinitely long thin shell subject to a pinched load is given in Buchwald (p. 246 of Reference 50) among others. The deflections under the load and at $\pi/2$ away from it are given by:

$$W(o) = \frac{4}{\pi}\,[3(1-\nu^2)]^{3/4}\left(\frac{R}{h}\right)^{3/2}\frac{P}{Eh}\sum_{k=1}^{\infty}\frac{k}{(4k^2-1)^{3/2}}$$

$$= 0.804168\,(1-\nu^2)^{3/4}\left(\frac{R}{h}\right)^{3/2}\frac{P}{Eh}$$

$$W(\tfrac{\pi}{2}) = \frac{4}{\pi}\,[3(1-\nu^2)]^{3/4}\left(\frac{R}{h}\right)^{3/2}\frac{P}{Eh}\sum_{k=1}^{\infty}\frac{(-1)^k k}{(4k^2-1)^{3/2}}$$

$$= -0.486281\,(1-\nu^2)^{3/4}\left(\frac{R}{h}\right)^{3/2}\frac{P}{Eh}$$

Finally a FORTRAN program has been developed for the VAX 11/780 computer to obtain the maximum central deflection of a simply supported cylindrical shell subject to a radial point load by the methods of Odqvist,[30] Bijlaard,[36] Mizoguchi,[38] and Yuan-Ting[25] as mentioned previously. In this program the algorithm for any cubic equation, based on the standard trigonometric method,[55] was developed and incorporated, in preference to approximate or iterative methods such as those shown in References 23, 45, 5, 66, 67, and 78, which in the precomputer era were popular. Furthermore, the inextensional solution of Timoshenko[4] for the pinched cylinder case at midlength with free ends has also been included.

CHAPTER 4

NUMERICAL SOLUTIONS

The models, which were studied in this report, were discretized by using a rectangular grid of points, when the elements were "plate elements," or a curved rectangular grid, when "shell elements" were employed.

POINT LOAD CASE

For STAGSC the discretization was carried out employing "410" plate quadrilateral elements. Their characteristics are described in Reference 2. For ABAQUS the modeling was carried out by using 8-noded "S8R" shell elements with reduced integration.[2] In all cases only half of the shell in axial length was analyzed.

There were two models used with both STAGSC and ABAQUS discretization. The first one (90°) was employed to investigate whether a simplification in halving the peripheral number of points can be made at the expense of accuracy. The second one (180°) is theoretically correct.

The discretization for STAGSC, where the first number indicates number of axial nodes and the second number of peripheral nodes, were:

5 x 13 for an angle of 90°
5 x 25 for an angle of 180°

For ABAQUS, due to midside nodes, the employed grids were:

9 x 25 for an angle of 90°.
9 x 49 for an angle of 180°.

The following boundary conditions were enforced:

At the simply supported	side x=0 :	V = W = RU = 0.*
At midlength (x = L/2)	symmetry :	U = RV = RW = 0.

---

* U = Displacement along the local and global axial axes (coincident)
V = Displacement along the local tangential axis
W = Displacement along the local radial axis
RU = Rotation about the local and global axial axes (coincident)
RV = Rotation about the local tangential axis
RW = Rotation about the local radial axis

At $\varphi = 0°$ (top generatrix) symmetry : V = RU = RW = 0.
At either $\varphi = 90°$ or 180° symmetry : V = RU = RW = 0.
All rigid body modes were suppressed.

PINCHED LOAD CASE

For STAGSC the modeling was carried out using "410" plate elements, while for ABAQUS through "S8R" shell elements. Again only half of the length of the shell axially was modeled. Only one type of discretization was employed in this case. It is given below for both STAGSC and ABAQUS.

STAGSC discretization: 5 x 13 for an angle of 90°.
ABAQUS discretization: 9 x 25 for an angle of 90°.

The following boundary conditions were applied:

At the free end (x=0) : RV = RW = 0.
At midlength (x=L/2) symmetry : U = RV = RW = 0.
At $\varphi = 90°$ (top generatrix) symmetry : V = RU = RW = 0.
All rigid body modes were suppressed.

CHAPTER 5

COMPARISON OF NUMERICAL AND ANALYTICAL SOLUTIONS

The objective of this work is, while trying to bring together some of the analytical solutions in the open literature, to further validate the two finite element programs STAGS[1] and ABAQUS[2] in the elastic range. For this purpose, a series of ten models each with a radius R=100 inches, thickness h=1.0 inch, and total length L ranging from 100.0 to 515.98 inches has been analyzed. Table 3 displays all geometrical material properties. Tables 4, 5, 8, 9, 10, 13, 15, and 17 pertain to the point load case (simple supports, V = W = RU = 0). Tables 6, 7, 11, 12, 14, and 16 contain results for the pinched load case (continuous supports, i.e., RV = RW = 0).

POINT LOAD CASE

Table 4 summarizes results of STAGS[1] analyses for a simply supported shell by modeling only a 90° segment peripherally and half length axially. Although we know that this is not exactly rigorous, at least for static analysis, we would like to examine its effect on the maximum central deflection.

Table 5 displays STAGS[1] results by modeling the correct 180° segment (Figure 14). It can be seen that, as L/R increases, the two answers start deviating by as much as 4.5 percent.

Table 9 displays ABAQUS[2] results for simply supported shells subject to a point force radially inwards by modeling a 90° segment (Figure 15) and should be compared to those of Table 4 (STAGS values). Table 10 contains ABAQUS results for a simply supported shell, by modeling the correct 180° segment and should be compared to Table 5 (STAGS values). The STAGS results for both cases are lower (stiffer model) than those by ABAQUS. In fact, Figure 12 (Case of R/h = 100) displays this feature clearly. We may observe here that ABAQUS employs Koiter-Sanders theory with a discretely imposed Kirchhoff assumption through a penalty method formulation. STAGS on the other hand uses a first order shell theory applicable to moderate rotations. The differences apparently must be attributed to the differences between flat (STAGS) versus curved shell elements (ABAQUS) with reduced integration and not so much to the formulation since we are dealing with "linear solutions."

Tables 13 and 15 contain analytical results for the range of radius/thickness (R/h) of 100 to 207.36 and radius/length (R/L) from 0.0969 to 1.0. Table 17 covers the range of R/h from 200 to 414.7 and R/L from 0.1938 to 1.0.

Figures 3 to 6, and 8 are plots of results of Tables 13 and 17, respectively, in terms of EWH/P versus R/L for various R/h ratios. Figures 9 and 10 are extensions of Figures 5 and 6, respectively, over R/h ratios from 200 to 414.72. Finally Figure 12, which applies to the case of R/h=100, summarizes the various analytical and numerical answers for the simply supported case. Generally, computations by STAGS yielded a stiffer model. The apparent increasing disagreement for ABAQUS results (Figure 12) from the analytical solutions as L/R increases are due to discretization. In all cases, the same mesh was used; it must be stressed, however, that the ABAQUS elements with their midside nodes increased the problem size considerably, while increasing accuracy.

PINCHED LOAD CASE

Table 6 displays pinched load results by STAGS. Tables 7 and 8 are two convergence studies for both the pinched cylinder and the simply supported cylinder cases. The two answers do not vary significantly.

Table 11 reports ABAQUS results for the pinched cylinder problem and must be compared to Table 6 (STAGS).

Tables 12 and 14 give the analytical results for R/h from 100.0 to 207.36, and R/L from 0.0969 to 1.000. Table 16 extends the range for R/h from 200.0 to 414.7 and R/L from 0.1938 to 1.000. Figures 7 and 11 show curves corresponding to the pinched cylinder problem, as given by Tables 12 and 16 respectively. Finally Figure 13, which applies to the case of R/h=100, summarizes the various analytical and numerical answers for the pinched case. For this case (Figure 13) both STAGS and ABAQUS produced larger displacements than the analytical solution. Here STAGS was closer to theory than ABAQUS.

Based on comparisons with the computed analytical solutions, ABAQUS performed better for the simply supported case, while STAGS performed better for the pinched cylinder problem. Naturally it would have been much better if the previous computational and analytical answers were verified by experiments, which is the ultimate check. What has emerged from the above analyses is the totally different behavior for the two types of support conditions and loadings (simply supported versus pinched cylinders) attributed to the membrane action even in the elastic range. The implication is this, although the behavior is different, for the same R/h as L/R increases, it "appears" that both solutions tend to the same value. However, the point load case approaches from below some asymptotic value, while the pinched load case, as can be seen from Equation (13), approaches zero asymptotically from above, which is not physically correct. Seide's solution[29] addresses the infinitely loaded cylinder subject to pinched radial loads. From the practical point of view, however, simplification of a problem to continuous supports from simple supports cannot occur, except in the case of a pipeline, which is indeed a long shell.

CONCLUSION

This study brings together existing analytical solutions for the cases of (1) a simply supported linear elastic circular cylindrical shell subject to a radially inwards concentrated force at midspan and (2) a linear elastic pinched cylindrical shell at midspan with free ends. It compares the analytical solutions with finite element computations by both STAGS[1] and ABAQUS[2] general purpose computer programs. Agreement is fairly good for both codes. While the ABAQUS answers are nearer to the analytical answers, it must be added that computations by STAGS were substantially faster. Furthermore, for static analyses, where the characteristic length circumferentially (such as 2R for a 180° modeling, versus R for a 90° segment) does not affect the calculation (for frequency or shock analyses this is not true), it has been shown by these simple calculations, that the effect of the load is indeed very localized (as demonstrated by the effect of the characteristic roots of the homogeneous equation).[5] As far as maximum displacement (in the elastic range) at the point of application is concerned, it is sufficient to model a 90° segment only. This is of importance to structural designers, as it reduces computational effort substantially.

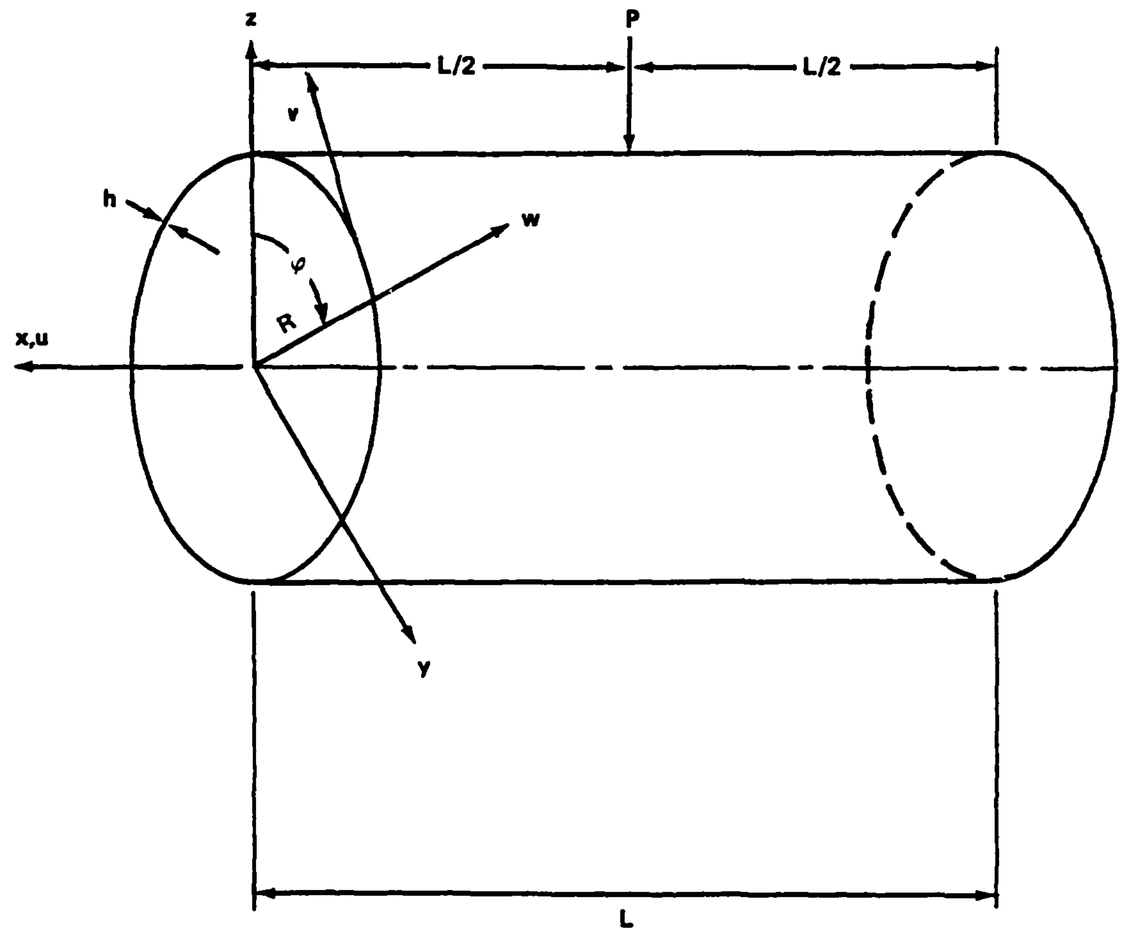


FIGURE 1. CYLINDRICAL SHELL SIMPLY SUPPORTED AT BOTH ENDS AND LOADED BY A RADIALLY INWARDS CONCENTRATED FORCE P . SHELL IS OF RADIUS R, THICKNESS h, AND LENGTH L

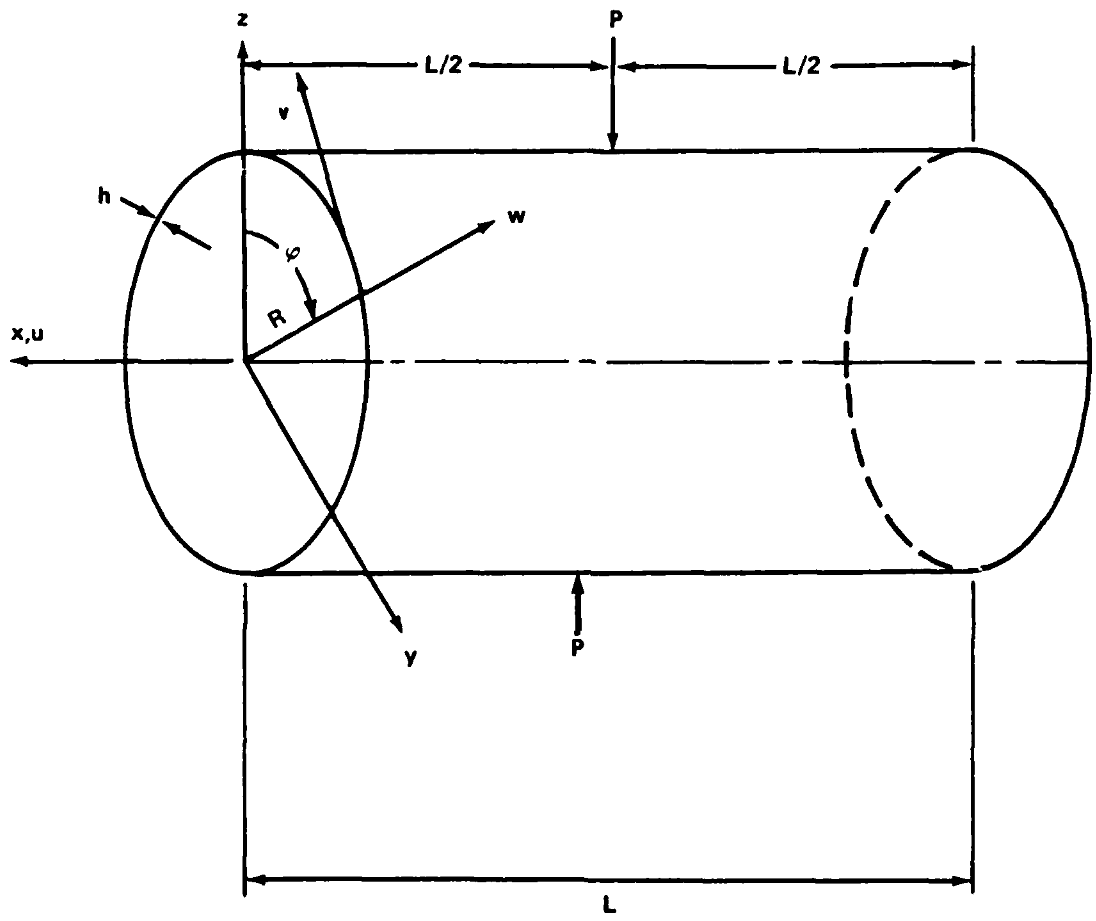


FIGURE 2. PINCHED CYLINDRICAL SHELL BY INWARDS CONCENTRATED FORCES P. SHELL IS OF RADIUS R, THICKNESS h, AND LENGTH L

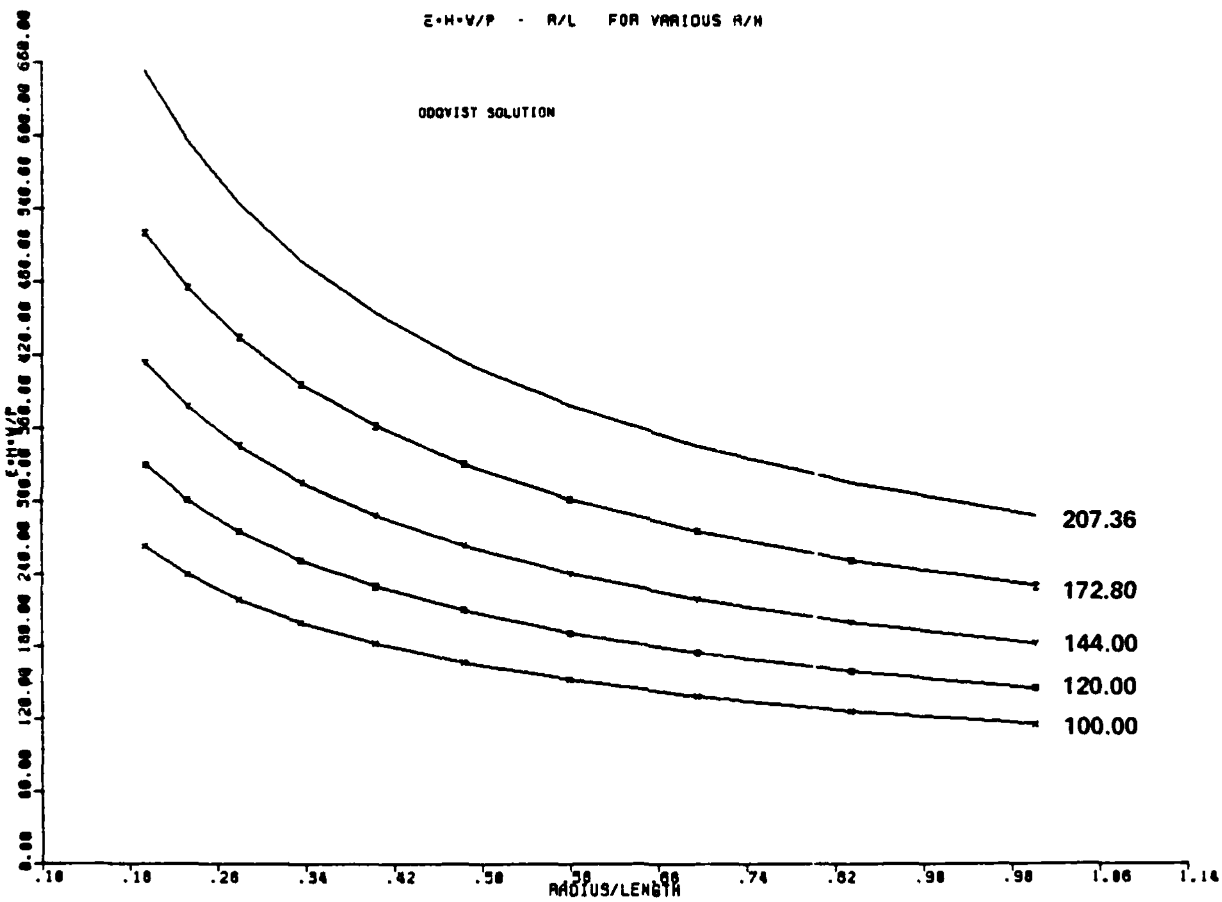


FIGURE 3. NONDIMENSIONAL PARAMETER EHW/P FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A RADIALLY INWARDS CENTRALLY LOCATED CONCENTRATED FORCE P (BY ODQVIST SOLUTION) PLOTTED VERSUS R/L FOR VARIOUS R/H RATIOS

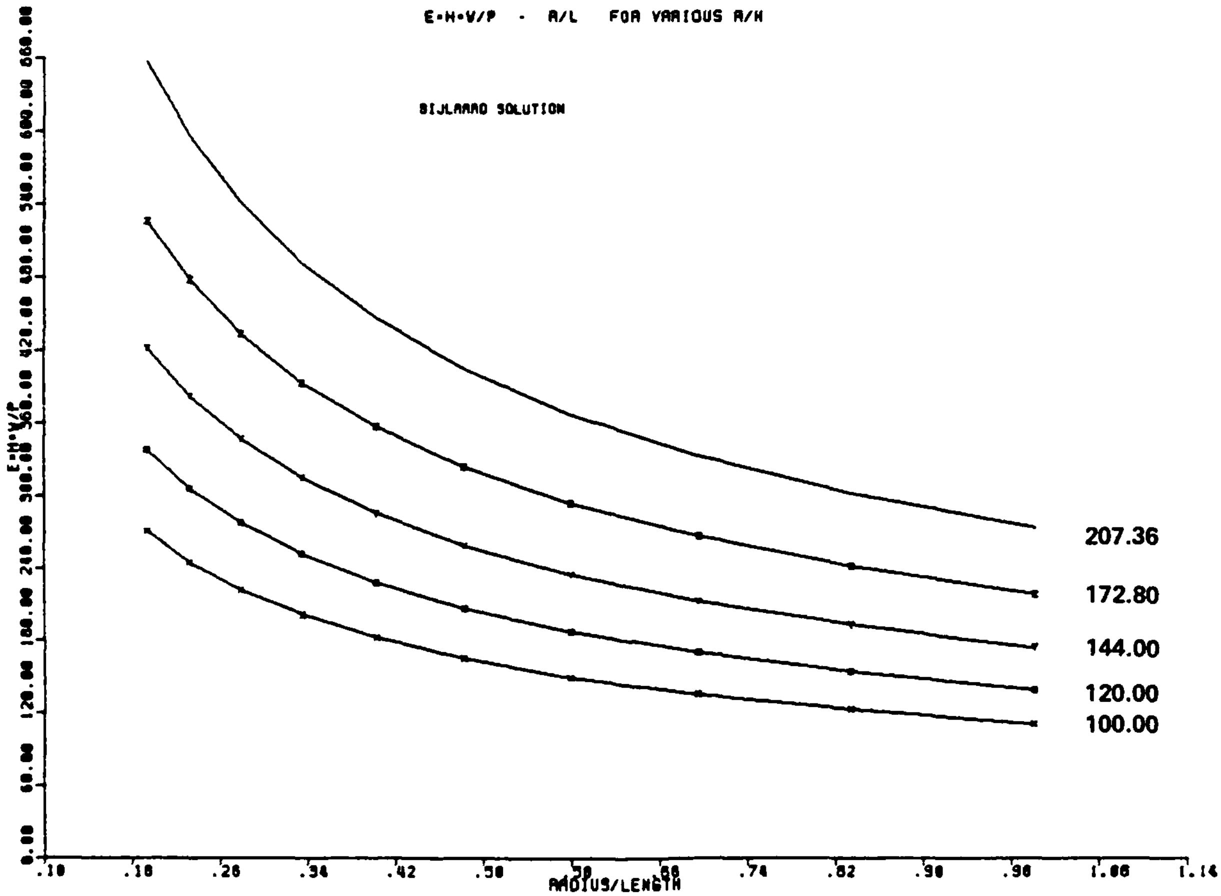


FIGURE 4. NONDIMENSIONAL PARAMETER EHW/P FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A RADIALLY INWARDS CENTRALLY LOCATED CONCENTRATED FORCE P (BY BIJLAARD SOLUTION) PLOTTED VERSUS R/L FOR VARIOUS R/H RATIOS

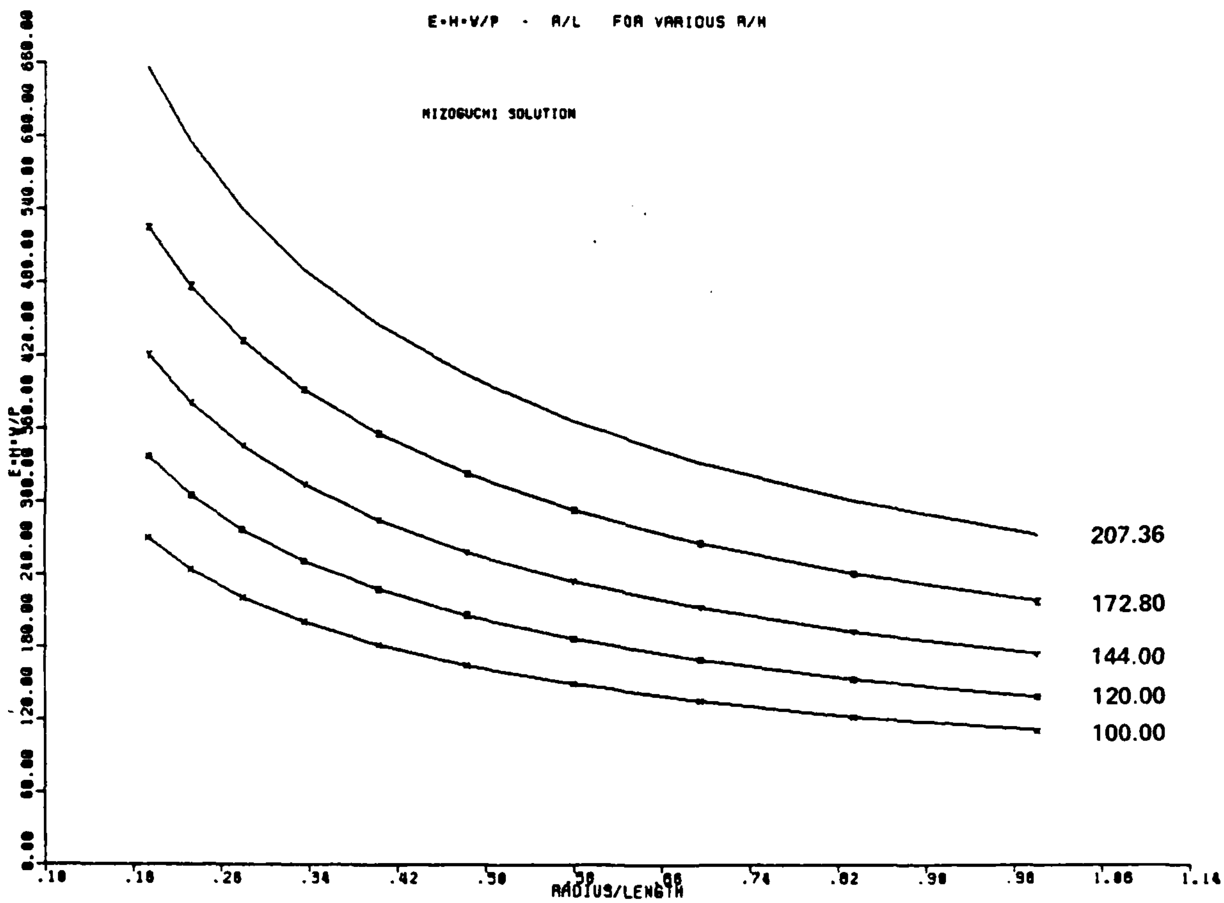


FIGURE 5. NONDIMENSIONAL PARAMETER EHW/P FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A RADIALLY INWARDS CENTRALLY LOCATED CONCENTRATED FORCE P (BY MIZOGUCHI SOLUTION) PLOTTED VERSUS R/L FOR VARIOUS R/H RATIOS

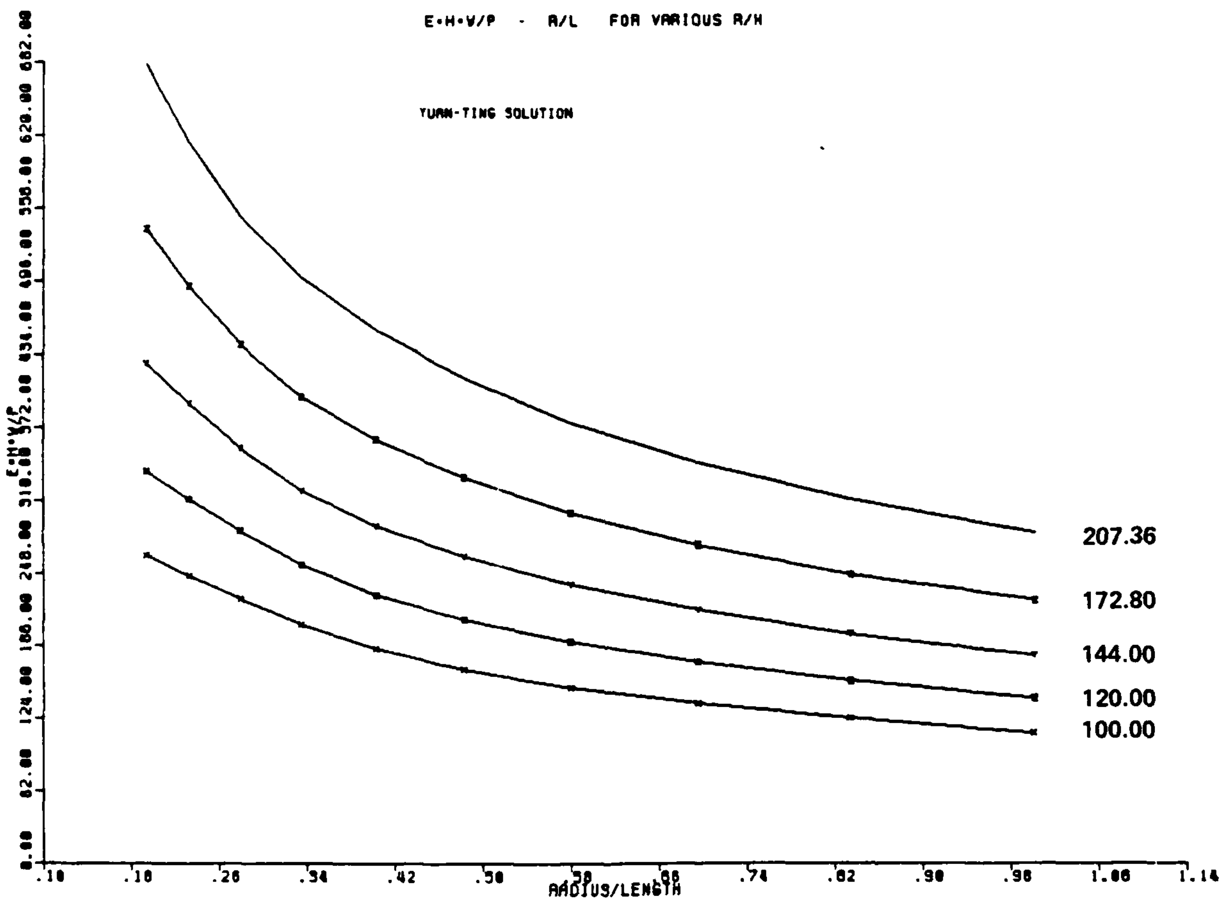


FIGURE 6. NONDIMENSIONAL PARAMETER EHW/P FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A RADIALLY INWARDS CENTRALLY LOCATED CONCENTRATED FORCE P (BY YUAN-TING SOLUTION) PLOTTED VERSUS R/L FOR VARIOUS R/H RATIOS

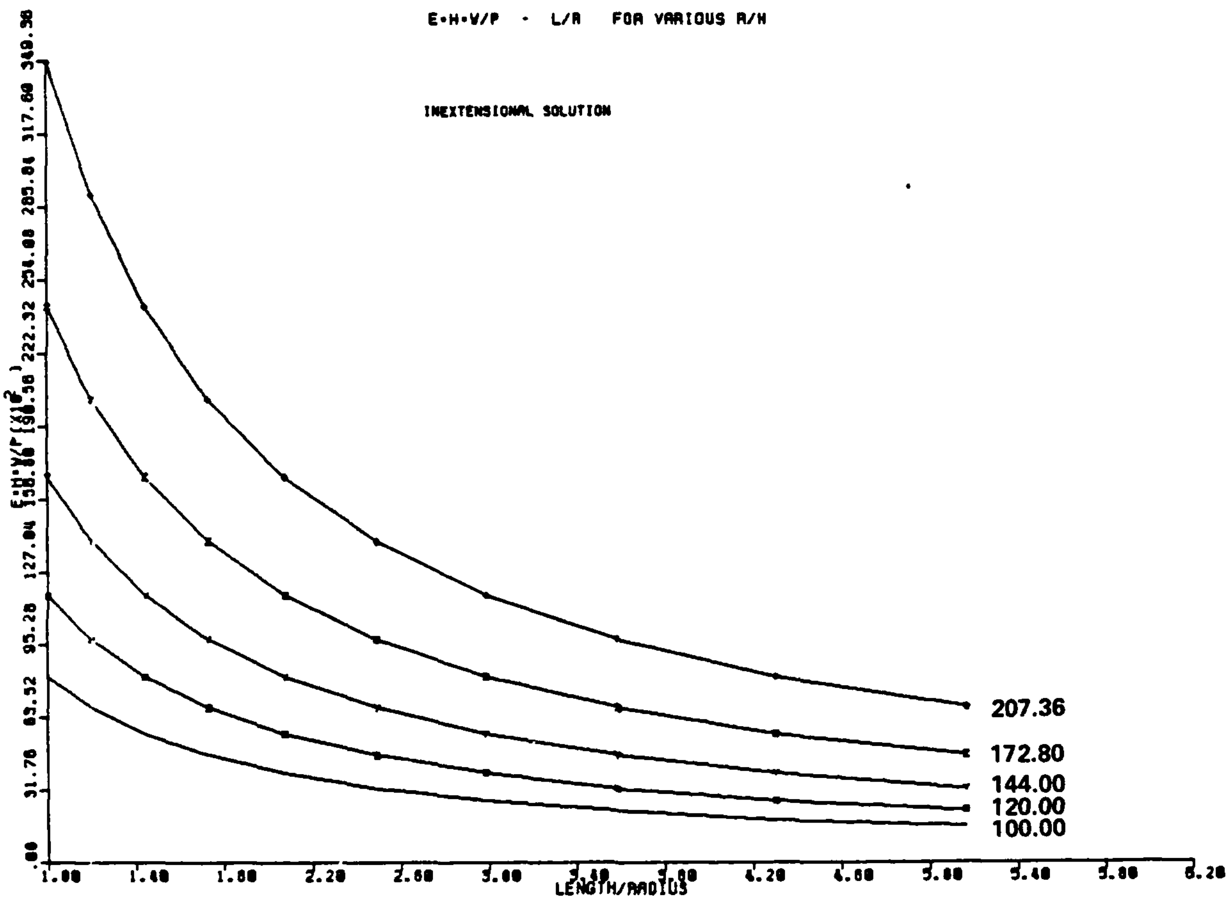


FIGURE 7. NONDIMENSIONAL PARAMENTER EHW/P FOR A RADIALLY INWARDS PINCHED CYLINDER WITH FREE ENDS (BY INEXTENSIONAL SOLUTION) PLOTTED VERSUS L/R FOR VARIOUS R/H RATIOS

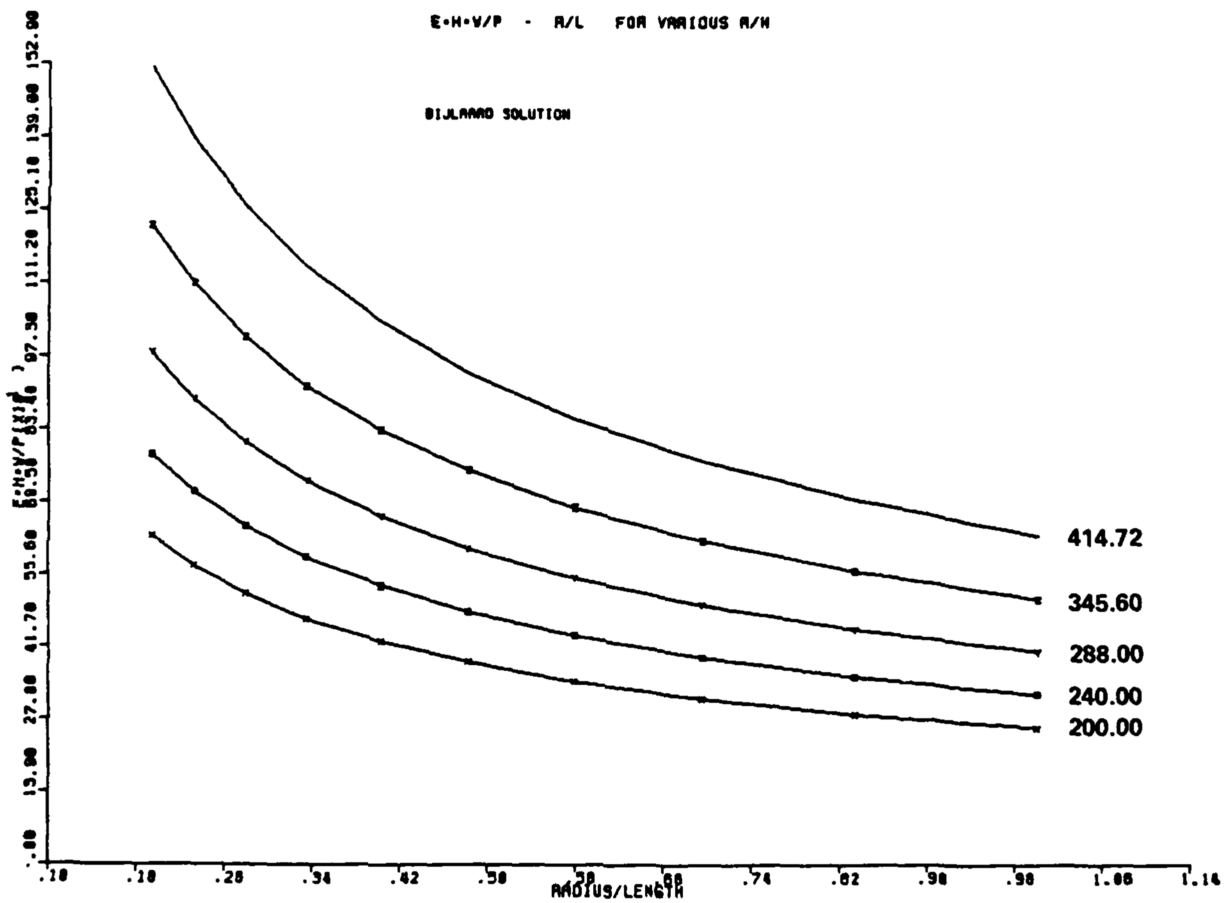


FIGURE 8. NONDIMENSIONAL PARAMETER EHW/P FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A RADIALLY INWARDS CENTRALLY LOCATED CONCENTRATED FORCE P (BY BIJLAARD SOLUTION) PLOTTED VERSUS R/L FOR VARIOUS R/H RATIOS

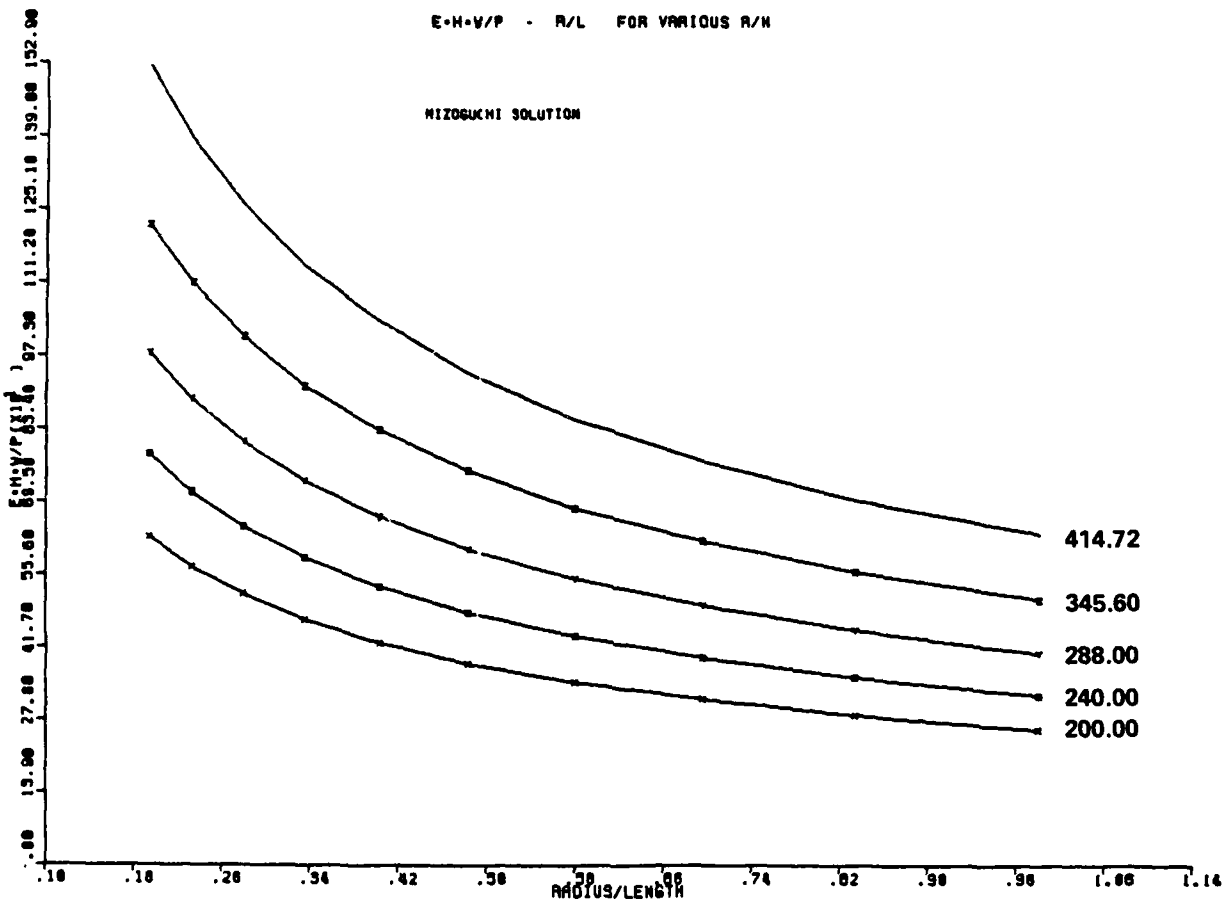


FIGURE 9. NONDIMENSIONAL PARAMETER EHW/P FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A RADIALLY INWARDS CENTRALLY LOCATED CONCENTRATED FORCE P (BY MIZOGUCHI SOLUTION) PLOTTED VERSUS R/L FOR VARIOUS R/H RATIOS

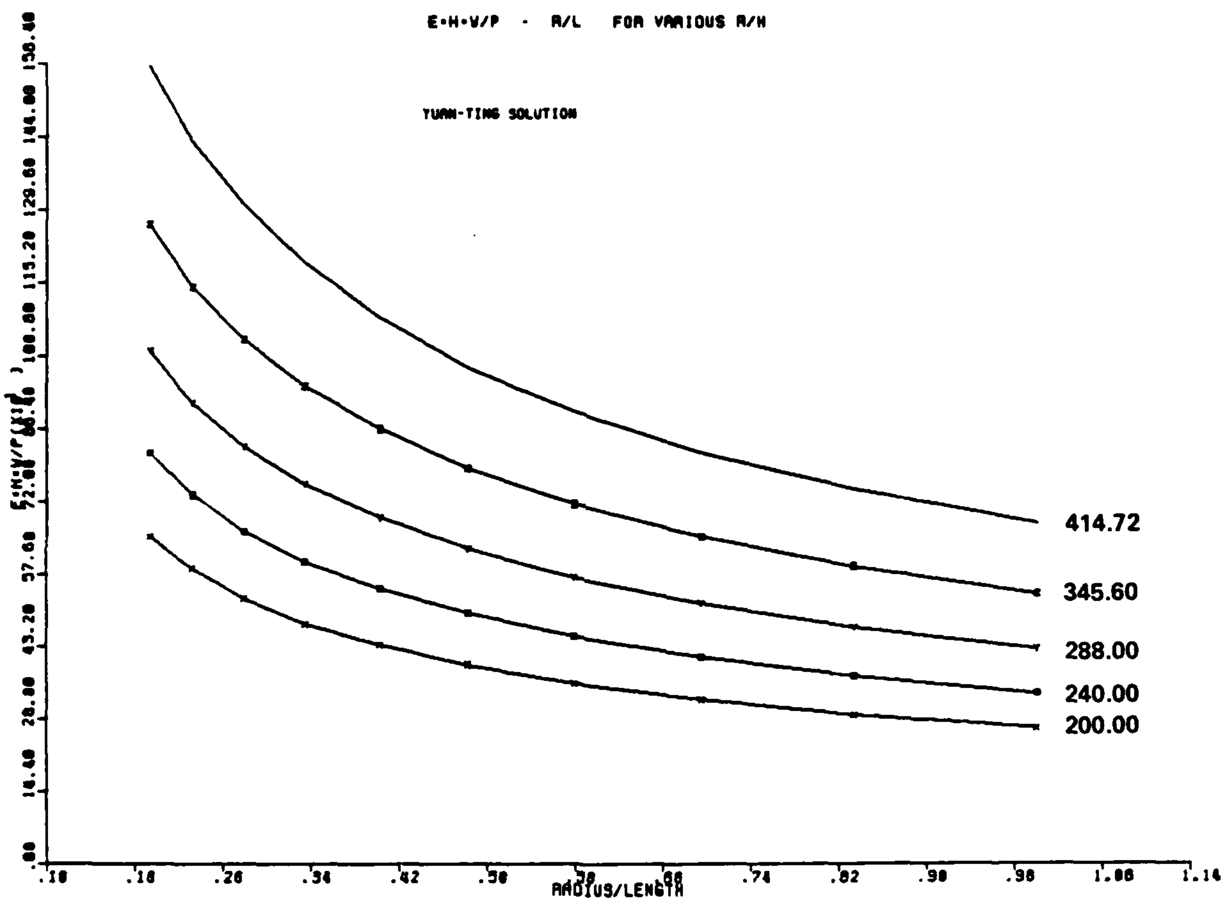


FIGURE 10. NONDIMENSIONAL PARAMETER EHW/P FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A RADIALLY INWARDS CENTRALLY LOCATED CONCENTRATED FORCE P (BY YUAN-TING SOLUTION) PLOTTED VERSUS R/L FOR VARIOUS R/H RATIOS

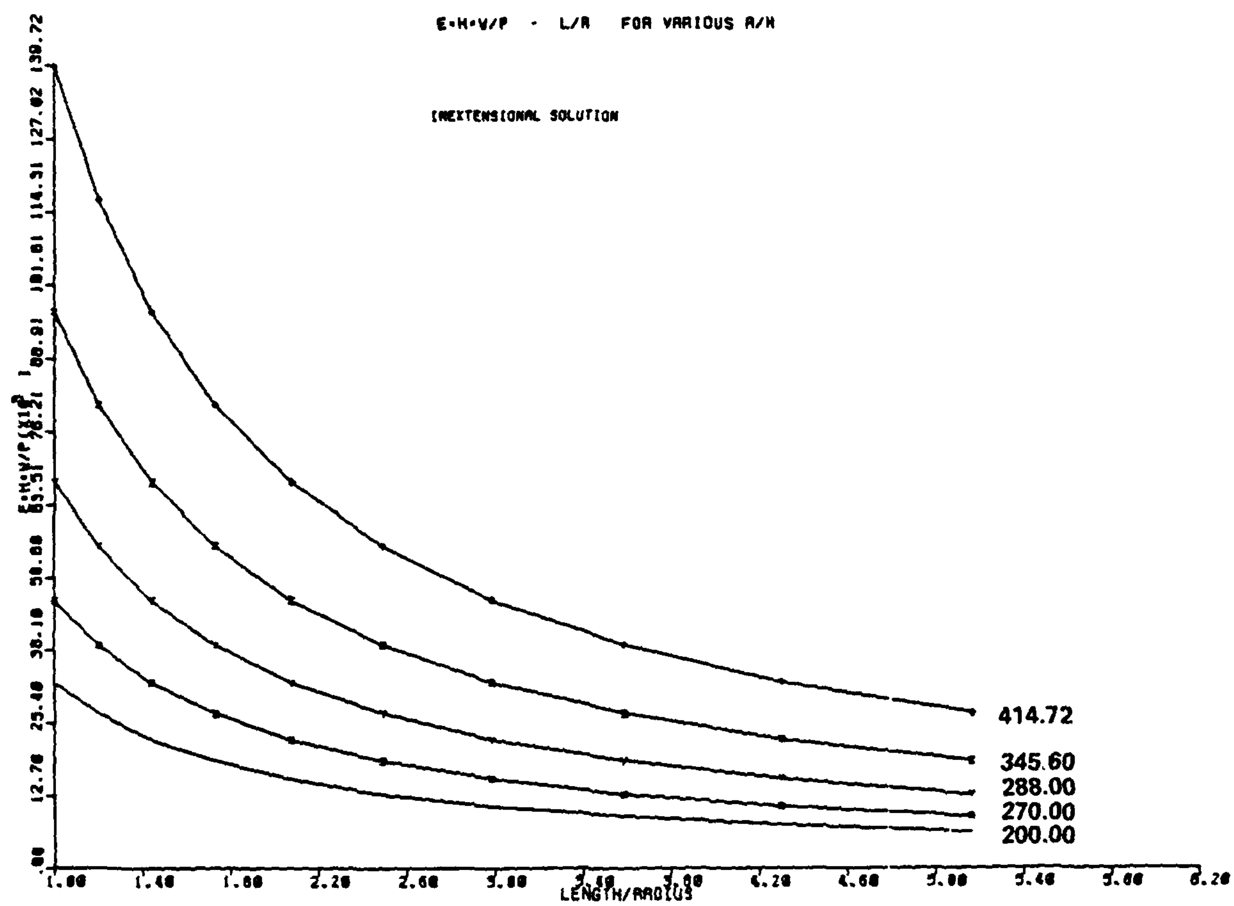
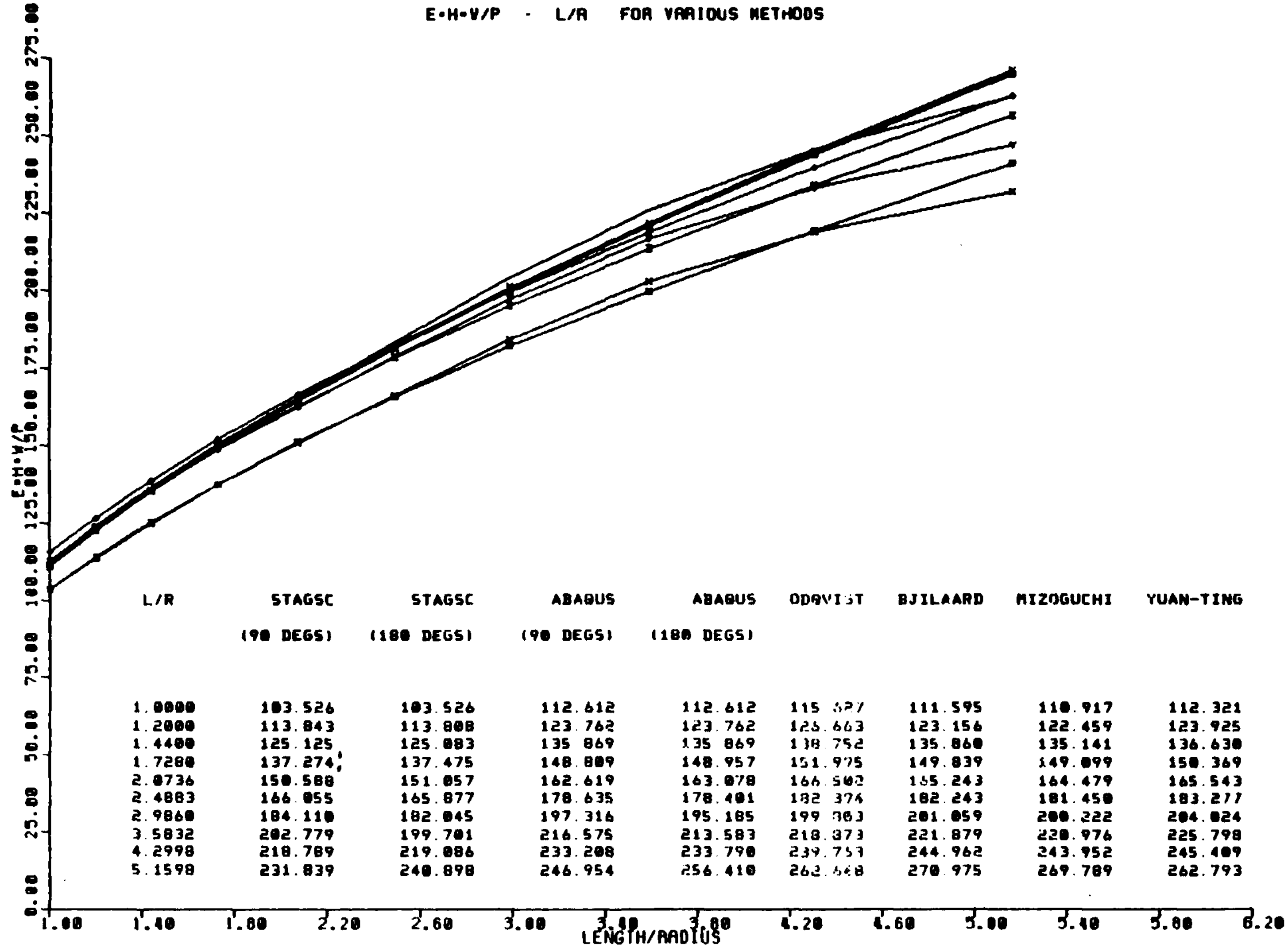
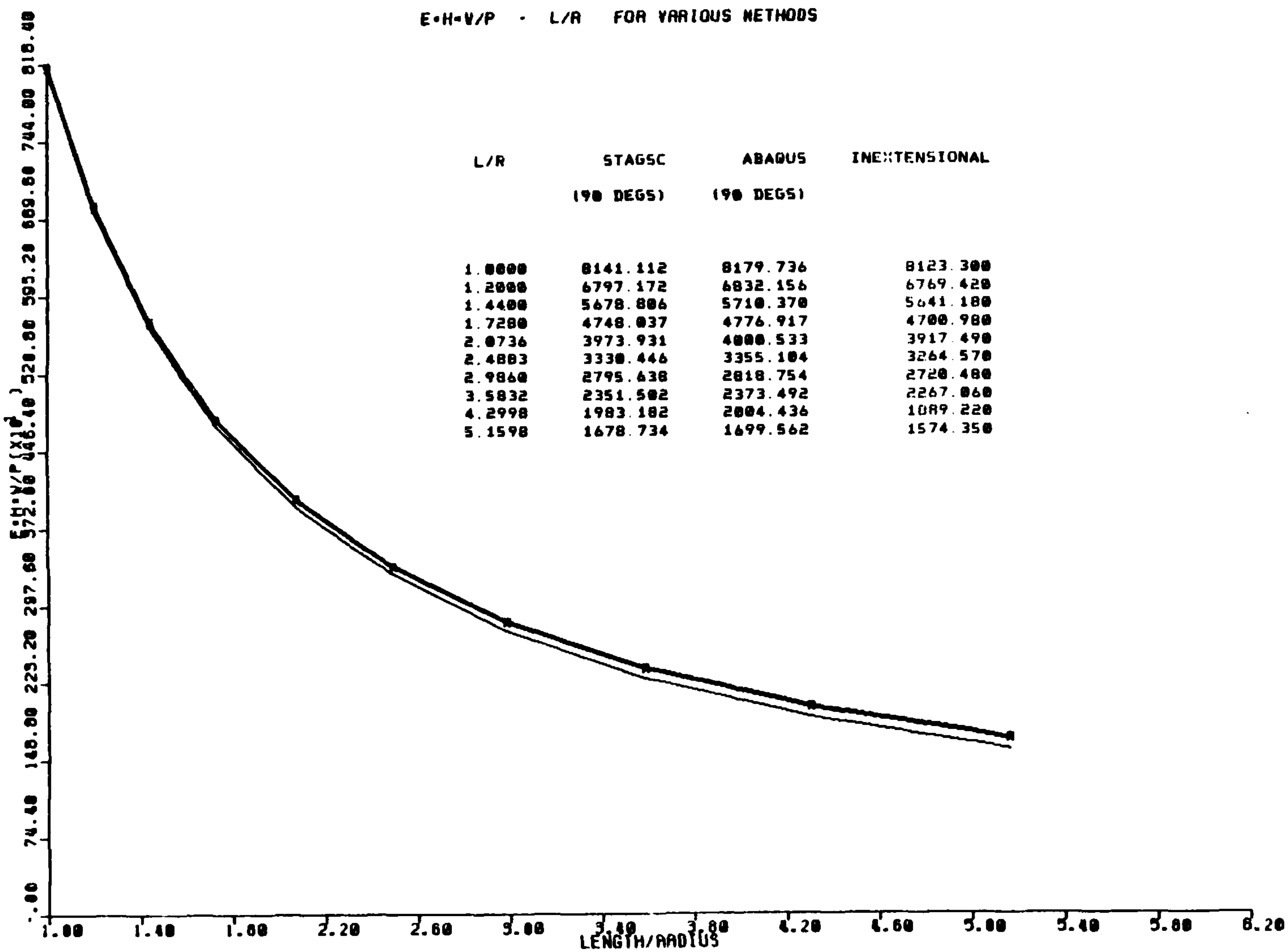


FIGURE 11. NONDIMENSIONAL PARAMETER EHW/P FOR A RADIALLY INWARDS PINCHED CYLINDER WITH FREE ENDS (BY INEXTENSIONAL SOLUTION) PLOTTED VERSUS L/R FOR VARIOUS R/H RATIOS



L/R	STAGSC (90 DEGS)	STAGSC (180 DEGS)	ABAQUS (90 DEGS)	ABAQUS (180 DEGS)	ODQVIST	BJILAARD	MIZOGUCHI	YUAN-TING
1.0000	103.526	103.526	112.612	112.612	115.527	111.595	110.917	112.321
1.2000	113.843	113.808	123.762	123.762	125.663	123.156	122.459	123.925
1.4400	125.125	125.083	135.869	135.869	138.752	135.860	135.141	136.630
1.7280	137.274	137.475	148.809	148.957	151.975	149.839	149.099	150.369
2.0736	150.588	151.057	162.619	163.078	166.502	165.243	164.479	165.543
2.4883	166.055	165.877	178.635	178.401	182.374	182.243	181.450	183.277
2.9860	184.110	182.045	197.316	195.185	199.383	201.059	200.222	204.024
3.5832	202.779	199.701	216.575	213.583	218.373	221.879	220.976	225.798
4.2998	218.789	219.086	233.208	233.790	239.757	244.962	243.952	245.409
5.1598	231.839	240.898	246.954	256.410	262.648	270.975	269.789	262.793

FIGURE 12. SUPERIMPOSED PLOTS OF ANALYTICAL SOLUTIONS BY FOUR METHODS AND FOUR FINITE ELEMENT DISCRETIZATIONS OF EHW/P VERSUS L/R FOR R/H = 100



L/R	STAGSC (90 DEGS)	ABAQUS (90 DEGS)	INEXTENSIONAL
1.0000	8141.112	8179.736	8123.300
1.2000	6797.172	6832.156	6769.420
1.4400	5678.806	5710.370	5641.180
1.7280	4748.037	4776.917	4700.980
2.0736	3973.931	4000.533	3917.490
2.4883	3330.446	3355.104	3264.570
2.9860	2795.638	2818.754	2720.480
3.5832	2351.502	2373.492	2267.060
4.2998	1983.182	2004.436	1089.220
5.1598	1678.734	1699.562	1574.350

FIGURE 13. SUPERIMPOSED PLOTS FOR A PINCHED CYLINDER BY TIMOSHENKO'S INEXTENSIONAL SOLUTION AND TWO FINITE ELEMENT DISCRETIZATIONS

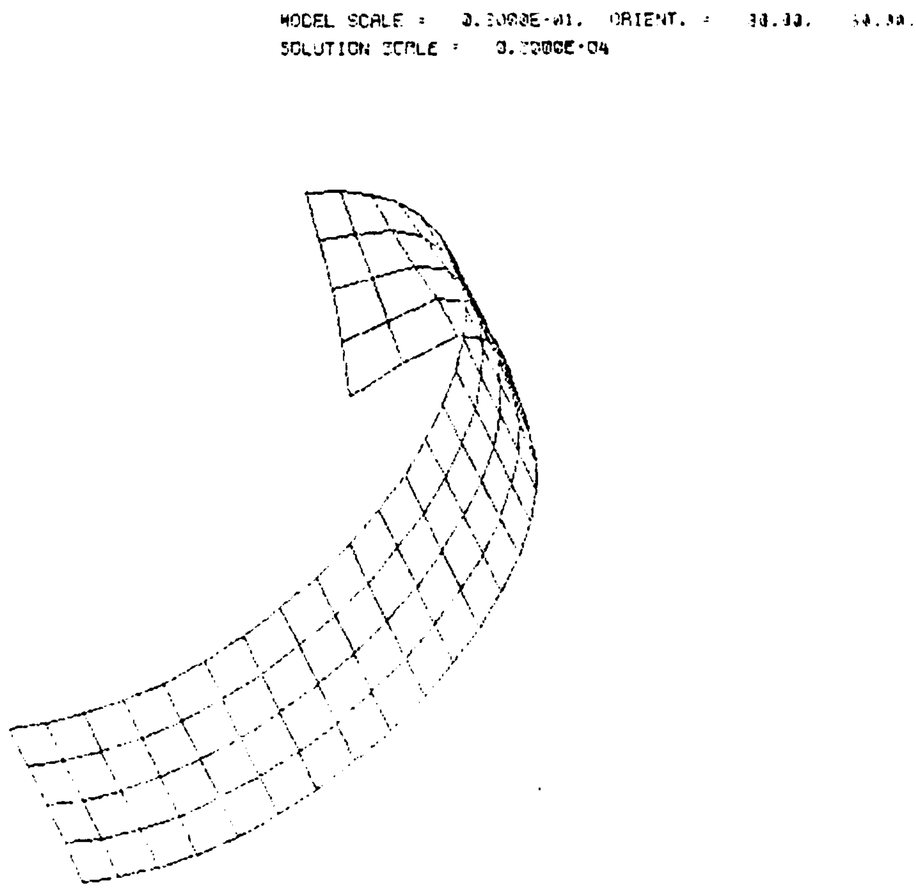


FIGURE 14. SIMPLY SUPPORTED CYLINDER ($\theta = 180^{\circ}$) SUBJECTED TO RADIAL CONCENTRATED LOAD (BY STAGS)

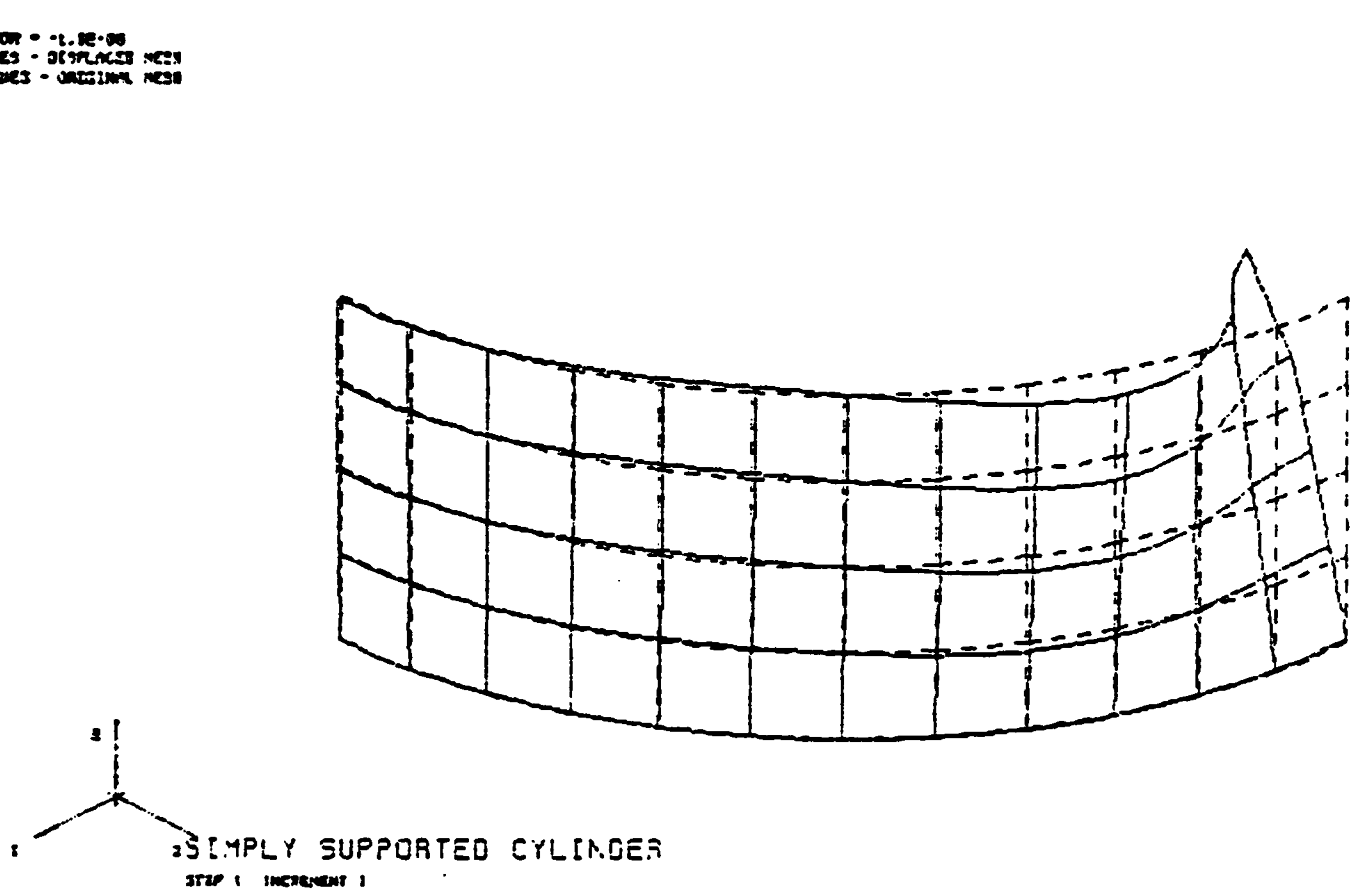


FIGURE 15. SIMPLY SUPPORTED CYLINDER (ℓ = 72", R = 100", θ = 90°) SUBJECTED TO RADIAL CONCENTRATED LOAD VIEWED FROM (100", 100", 100") (BY ABAQUS)

TABLE 1. GOVERNING DIFFERENTIAL EQUATIONS USED FOR ANALYTICAL SOLUTIONS*

AUTHOR	DIFFERENTIAL EQUATION SOLVED
ODQVIST[30]	$\frac{\partial^8 M_\varphi}{\partial\varphi^8} + \frac{12(1-\nu^2)a^2}{h^2}\frac{\partial^4 M_\varphi}{\partial x^4} = 0$
BIJLAARD[36]	$\nabla^8 w + \frac{12(1-\nu^2)}{R^2h^2}\frac{\partial^4 w}{\partial x^4} + \frac{1}{R^2}\left[\frac{2}{R^6}\frac{\partial^6 w}{\partial\varphi^6} + \frac{(\nu+7)}{R^4}\frac{\partial^6 w}{\partial x^2\partial\varphi^4} + \frac{(6+\nu-\nu^2)}{R^2}\frac{\partial^6 w}{\partial x^4\partial\varphi^2}\right] - \frac{1}{D}\nabla^4 p_r = 0$
MIZOGUCHI[38]	$A\frac{\partial^8 w}{\partial x^8} + \frac{4B}{R^2}\frac{\partial^8 w}{\partial x^6\partial\varphi^2} + \frac{6C}{R^4}\frac{\partial^8 w}{\partial x^4\partial\varphi^4} + \frac{8-2\nu^2}{R^4}\frac{\partial^6 w}{\partial x^4\partial\varphi^2} + \frac{6L}{R^4}\frac{\partial^4 w}{\partial x^4} + \frac{4}{R^6}\frac{\partial^2}{\partial x^2}\left[\frac{\partial^6 w}{\partial\varphi^6} + 2\frac{\partial^4 w}{\partial\varphi^4} + \frac{\partial^2 w}{\partial\varphi^2}\right] + \frac{1}{R^8}\frac{\partial^2}{\partial\varphi^2}\left[\frac{\partial^6 w}{\partial\varphi^6} + 2\frac{\partial^4 w}{\partial\varphi^4} + \frac{\partial^2 w}{\partial\varphi^2}\right] = \frac{1}{D}\left[A\frac{\partial^4 p}{\partial x^4} + \frac{2U}{R^2}\frac{\partial^4 p}{\partial x^2\partial\varphi^2} + \frac{B}{R^4}\frac{\partial^4 p}{\partial\varphi^4}\right]$ $A = 1 + \frac{1}{3}\left(\frac{h}{R}\right)^2, \quad B = 1 + \frac{1}{12}\left(\frac{h}{R}\right)^2, \quad C = 1 + \frac{1-\nu^2}{72}\left(\frac{h}{R}\right)^2$ $L = 2(1-\nu^2)\left(\frac{R}{h}\right)^2 A, \quad U = 1 + \frac{1+(1-\nu)^2}{12(1-\nu)}\left(\frac{h}{R}\right)^2$
YUAN-TING[25]	$\nabla^8 w + \frac{12(1-\nu^2)}{R^2h^2}\frac{\partial^4 w}{\partial x^4} + \frac{2}{R^2}\frac{\partial^6 w}{\partial s^6} + 2(4-\nu)\frac{\partial^6 w}{\partial s^4\partial x^2} + 6\frac{\partial^6 w}{\partial s^2\partial x^4} + 2\nu\frac{\partial^6 w}{\partial x^6} + \frac{1}{R^4}\frac{\partial^4 w}{\partial s^4} + 2(2-\nu)\frac{\partial^4 w}{\partial s^2\partial x^2} - \frac{1}{D}\nabla^4 q = 0$
TIMOSHENKO[4]	Energy Argument Through Principle of Virtual Displacements

* For definition of employed variables see Nomenclature

TABLE 2. SUMMARY OF SOME PRACTICAL METHODS

AUTHOR	REF	EQUATION TYPE	PROBLEM TYPE	BOUNDARY CONDITIONS	USED METHOD
ODQVIST	30	8th order P.D.E. in B. Moment M_φ as simplified by Schorer	Uniform load along cylinder's Generatrix. In the limit radial point load.	Simply supported at both ends	Fourier Series axially with linear combination of decaying exponential and trigonometric functions circumferentially
BIJLAARD	36	8th order P.D.E. in radial displacement w based on Timoshenko type of equations with simplifications.	Unform load along a rectangular element on cylinder's surface. Radial point load in the limit	Simply supported at both ends	Double Fourier series to match series representation of load and boundary conditions
MIZOGUCHI	38	8th order P.D.E. in radial displacement w	Radial concentrated load	Simply supported at both ends	Finite Fourier transform and method of images
YUAN AND TING	25	8th order P.D.E. in w based on Fluegge equations	Radial concentrated load	Simply supported at both ends	Fourier series and Fourier integral with method of images
TIMOSHENKO	4	Inextensional Theory for three strains at the middle surface requires them to be zero.	Radial Concentrated load	Finite length with free ends	Fourier series peripherally to satisfy inextensional condition. Use minimum potential energy.

TABLE 3. GEOMETRICAL AND MATERIAL PROPERTIES OF ANALYZED MODELS

MODEL NO.	RADIUS R (in.)	THICKNESS h (in.)	ANALYZED LENGTH L (in.)	TOTAL LENGTH L_T (in.)	R/h	L_T/R	YOUNG'S MODULUS E (Lb/in^2)	POISSONS RATIO ν
1	100.0	1.0	50.000	100.00	100	1.0000	30×10^6	0.300
2	100.0	1.0	60.000	120.00	100	1.2000	30×10^6	0.300
3	100.0	1.0	72.000	144.00	100	1.4400	30×10^6	0.300
4	100.0	1.0	86.400	172.80	100	1.7280	30×10^6	0.300
5	100.0	1.0	103.680	207.36	100	2.0736	30×10^6	0.300
6	100.0	1.0	124.415	248.83	100	2.4883	30×10^6	0.300
7	100.0	1.0	149.300	298.60	100	2.9860	30×10^6	0.300
8	100.0	1.0	179.160	358.32	100	3.5832	30×10^6	0.300
9	100.0	1.0	214.990	429.98	100	4.2998	30×10^6	0.300
10	100.0	1.0	257.990	515.98	100	5.1598	30×10^6	0.300

TABLE 4. RESULTS BY STAGSC COMPUTER PROGRAM FOR A SIMPLY SUPPORTED SHELL (V=W=RU=0) SUBJECT TO A CENTRALLY LOCATED INWARD CONCENTRATED FORCE

MODEL NO.	NO. OF NODES AXIALLY INCLUDING MIDSIDE NODES (IF ANY)	NO. OF NODES PERIPHERALLY INCLUDING MIDSIDE NODES (IF ANY)	ANALYZED SUBTENDED ANGLE (°)	MIDSIDE NODES	REACTION ON QUARTER MODEL P/4 (LB)	W RADIAL INWARD DISPLACEMENT (IN)	$\frac{Ehw}{P}$
1	5	13	90°	NO	-14.489	-0.0002	103.526
2	5	13	90°	NO	-13.176	-0.0002	113.843
3	5	13	90°	NO	-11.988	-0.0002	125.125
4	5	13	90°	NO	-10.927	-0.0002	137.274
5	5	13	90°	NO	- 9.9609	-0.0002	150.588
6	5	13	90°	NO	- 9.0331	-0.0002	166.055
7	5	13	90°	NO	- 8.1473	-0.0002	184.110
8	5	13	90°	NO	- 7.3972	-0.0002	202.779
9	5	13	90°	NO	- 6.8559	-0.0002	218.789
10	5	13	90°	NO	- 6.4700	-0.0002	231.839

TABLE 5. RESULTS BY STAGSC COMPUTER PROGRAM FOR A SIMPLY SUPPORTED SHELL (V=W=RU=0) SUBJECT TO A CENTRALLY LOCATED INWARD CONCENTRATED FORCE

MODEL NO.	NO. OF NODES AXIALLY INCLUDING MIDSIDE NODES (IF ANY)	NO. OF NODES PERIPHERALLY INCLUDING MIDSIDE NODES (IF ANY)	ANALYZED SUBTENDED ANGLE (°)	MIDSIDE NODES	REACTION ON QUARTER MODEL P/4 (LB)	W RADIAL INWARD DISPLACEMENT (IN)	$\frac{Ehw}{P}$
1	5	25	180°	NO	-14.489*	-0.0002	103.526
2	5	25	180°	NO	-13.180	-0.0002	113.808
3	5	25	180°	NO	-11.992	-0.0002	125.083
4	5	25	180°	NO	-10.911	-0.0002	137.475
5	5	25	180°	NO	- 9.930	-0.0002	151.057
6	5	25	180°	NO	- 9.0428	-0.0002	165.877
7	5	25	180°	NO	- 8.2397	-0.0002	182.045
8	5	25	180°	NO	- 7.5112	-0.0002	199.701
9	5	25	180°	NO	- 6.8466	-0.0002	219.086
10	5	25	180°	NO	- 6.2267	-0.0002	240.898

*All values were obtained with IPENL = 0 (penalty function on fourth order terms in element 410) in N-1 cards. If IPENL = 1 the first entry for P/4 becomes -13.238.

TABLE 6. RESULTS BY STAGSC COMPUTER PROGRAM FOR A PINCHED SHELL (BOUNDARY CONDITIONS AT ENDS RV=RW=0)

MODEL NO.	NO. OF NODES AXIALLY INCLUDING MIDSIDE NODES (IF ANY)	NO. OF NODES PERIPHERALLY INCLUDING MIDSIDE NODES (IF ANY)	ANALYZED SUBTENDED ANGLE (°)	MIDSIDE NODES	REACTION ON QUARTER MODEL P/4 (LB)	W RADIAL INWARD DISPLACEMENT (IN)	$\frac{Ehw}{P}$
1	5	13	90°	NO	-0.18425*	-0.0002	8141.112
2	5	13	90°	NO	-0.22068	-0.0002	6797.172
3	5	13	90°	NO	-0.26414	-0.0002	5678.806
4	5	13	90°	NO	-0.31592	-0.0002	4748.037
5	5	13	90°	NO	-0.37746	-0.0002	3973.931
6	5	13	90°	NO	-0.45039	-0.0002	3330.446
7	5	13	90°	NO	-0.53655	-0.0002	2795.638
8	5	13	90°	NO	-0.63789	-0.0002	2351.502
9	5	13	90°	NO	-0.75636	-0.0002	1983.182
10	5	13	90°	NO	-0.89353	-0.0002	1678.734

*All values were obtained with IPENL = 0 (penalty function on fourth order terms in element 410) in N-1 cards. If IPENL = 1 the first entry for P/4 becomes -0.18404.

TABLE 7. CONVERGENCE STUDY BY STAGSC COMPUTER PROGRAM OF MODEL NO. 1 WITH CONTINUITY END CONDITIONS

MODEL NO.	NO. OF NODES AXIALLY INCLUDING MIDSIDE NODES (IF ANY)	NO. OF NODES PERIPHERALLY INCLUDING MIDSIDE NODES (IF ANY)	ANALYZED SUBTENDED ANGLE (°)	MIDSIDE NODES	REACTION ON QUARTER MODEL P/4 (LB)	W RADIAL INWARD DISPLACEMENT (IN)	$\frac{Ehw}{P}$
1	5	13	90°	NO	-0.18425	-0.0002	8141.112
1	9	25	90°	NO	-0.18361	-0.0002	8169.489
1	19	51	90°	NO	-0.18343	-0.0002	8177.506

TABLE 8. CONVERGENCE STUDY BY STAGSC COMPUTER PROGRAM OF MODEL NO. 1 WITH SIMPLY SUPPORTED ENDS

MODEL NO.	NO. OF NODES AXIALLY INCLUDING MIDSIDE NODES (IF ANY)	NO. OF NODES PERIPHERALLY INCLUDING MIDSIDE NODES (IF ANY)	ANALYZED SUBTENDED ANGLE (°)	MIDSIDE NODES	REACTION ON QUARTER MODEL P/4 (LB)	W RADIAL INWARD DISPLACEMENT (IN)	$\frac{Ehw}{P}$
1	5	13	90°	NO	-14.489	-0.0002	103.526
1	9	25	90°	NO	-13.582	-0.0002	110.440
1	19	51	90°	NO	-12.044	-0.0002	124.543

TABLE 9. RESULTS BY ABAQUS COMPUTER PROGRAM FOR A SIMPLY SUPPORTED SHELL (V=W=RU=0) SUBJECT TO A CENTRALLY LOCATED INWARD CONCENTRATED FORCE

MODEL NO.	NO. OF NODES AXIALLY INCLUDING MIDSIDE NODES (IF ANY)	NO. OF NODES PERIPHERALLY INCLUDING MIDSIDE NODES (IF ANY)	ANALYZED SUBTENDED ANGLE (°)	MIDSIDE NODES	REACTION ON QUARTER MODEL P/4 (LB)	W RADIAL INWARD DISPLACEMENT (IN)	$\frac{Ehw}{P}$
1	9	25	90°	YES	-1.332	-2×10^{-5}	112.612
2	9	25	90°	YES	-1.212	-2×10^{-5}	123.762
3	9	25	90°	YES	-1.104	-2×10^{-5}	135.869
4	9	25	90°	YES	-1.008	-2×10^{-5}	148.809
5	9	25	90°	YES	-0.9224	-2×10^{-5}	162.619
6	9	25	90°	YES	-0.8397	-2×10^{-5}	178.635
7	9	25	90°	YES	-0.7602	-2×10^{-5}	197.316
8	9	25	90°	YES	-0.6926	-2×10^{-5}	216.575
9	9	25	90°	YES	-0.6432	-2×10^{-5}	233.208
10	9	25	90°	YES	-0.6074	-2×10^{-5}	246.954

TABLE 10. RESULTS BY ABAQUS COMPUTER PROGRAM FOR A SIMPLY SUPPORTED SHELL (V=W=RU=0) SUBJECT TO A CENTRALLY LOCATED INWARD CONCENTRATED FORCE

MODEL NO.	NO. OF NODES AXIALLY INCLUDING MIDSIDE NODES (IF ANY)	NO. OF NODES PERIPHERALLY INCLUDING MIDSIDE NODES (IF ANY)	ANALYZED SUBTENDED ANGLE (°)	MIDSIDE NODES	REACTION ON QUARTER MODEL P/4 (LB)	W RADIAL INWARD DISPLACEMENT (IN)	$\frac{Ehw}{P}$
1	9	49	180°	YES	-1.332	$2x10^{-5}$	112.612
2	9	49	180°	YES	-1.212	$2x10^{-5}$	123.762
3	9	49	180°	YES	-1.104	$2x10^{-5}$	135.869
4	9	49	180°	YES	-1.007	$2x10^{-5}$	148.957
5	9	49	180°	YES	-0.9198	$2x10^{-5}$	163.078
6	9	49	180°	YES	-0.8408	$2x10^{-5}$	178.401
7	9	49	180°	YES	-0.7685	$2x10^{-5}$	195.185
8	9	49	180°	YES	-0.7023	$2x10^{-5}$	213.583
9	9	49	180°	YES	-0.6416	$2x10^{-5}$	233.790
10	9	49	180°	YES	-0.5850	$2x10^{-5}$	256.410

TABLE 11. RESULTS BY ABAQUS COMPUTER PROGRAM FOR A PINCHED SHELL (BOUNDARY CONDITIONS AT ENDS RV=RW=0)

MODEL NO.	NO. OF NODES AXIALLY INCLUDING MIDSIDE NODES (IF ANY)	NO. OF NODES PERIPHERALLY INCLUDING MIDSIDE NODES (IF ANY)	ANALYZED SUBTENDED ANGLE (°)	MIDSIDE NODES	REACTION ON QUARTER MODEL P/4 (LB)	W RADIAL INWARD DISPLACEMENT (IN)	$\frac{Ehw}{P}$
1	9	25	90	YES	-1.8338×10^{-2}	-2×10^{-5}	8179.736
2	9	25	90	YES	-2.1955×10^{-2}	-2×10^{-5}	6832.156
3	9	25	90	YES	-2.6268×10^{-2}	-2×10^{-5}	5710.370
4	9	25	90	YES	-3.1401×10^{-2}	-2×10^{-5}	4776.917
5	9	25	90	YES	-3.7495×10^{-2}	-2×10^{-5}	4000.533
6	9	25	90	YES	-4.4708×10^{-2}	-2×10^{-5}	3355.104
7	9	25	90	YES	-5.3215×10^{-2}	-2×10^{-5}	2818.754
8	9	25	90	YES	-6.3198×10^{-2}	-2×10^{-5}	2373.492
9	9	25	90	YES	-7.4834×10^{-2}	-2×10^{-5}	2004.436
10	9	25	90	YES	-8.8258×10^{-2}	-2×10^{-5}	1699.562

TABLE 12. PARAMETER EWH/P FOR VARIOUS R/H AND R/L RATIOS FOR A PINCHED CYLINDRICAL SHELL BY TIMOSHENKO'S INEXTENSIONAL SOLUTION

E*W*H/P

R/H = 0.100000E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
100.000	1.000	100.00	0.100000E+01	0.812330E+04
100.000	1.000	120.00	0.833333E+00	0.676942E+04
100.000	1.000	144.00	0.694444E+00	0.564118E+04
100.000	1.000	172.80	0.578704E+00	0.470098E+04
100.000	1.000	207.36	0.482253E+00	0.391749E+04
100.000	1.000	248.83	0.401878E+00	0.326457E+04
100.000	1.000	298.60	0.334898E+00	0.272048E+04
100.000	1.000	358.32	0.279082E+00	0.226706E+04
100.000	1.000	429.98	0.232568E+00	0.188922E+04
100.000	1.000	515.98	0.193807E+00	0.157435E+04

R/H = 0.120000E+03

120.000	1.000	120.00	0.100000E+01	0.116976E+05
120.000	1.000	144.00	0.833333E+00	0.974796E+04
120.000	1.000	172.80	0.694444E+00	0.812330E+04
120.000	1.000	207.36	0.578704E+00	0.676942E+04
120.000	1.000	248.83	0.482253E+00	0.564118E+04
120.000	1.000	298.60	0.401878E+00	0.470098E+04
120.000	1.000	358.32	0.334898E+00	0.391749E+04
120.000	1.000	429.98	0.279082E+00	0.326457E+04
120.000	1.000	515.98	0.232568E+00	0.272048E+04
120.000	1.000	619.17	0.193807E+00	0.226706E+04

R/H = 0.144000E+03

144.000	1.000	144.00	0.100000E+01	0.168445E+05
144.000	1.000	172.80	0.833333E+00	0.140371E+05
144.000	1.000	207.36	0.694444E+00	0.116976E+05
144.000	1.000	248.83	0.578704E+00	0.974796E+04
144.000	1.000	298.60	0.482253E+00	0.812330E+04
144.000	1.000	358.32	0.401878E+00	0.676942E+04
144.000	1.000	429.98	0.334898E+00	0.564118E+04
144.000	1.000	515.98	0.279082E+00	0.470098E+04
144.000	1.000	619.17	0.232568E+00	0.391749E+04
144.000	1.000	743.01	0.193807E+00	0.326457E+04

R/H = 0.172800E+03

172.800	1.000	172.80	0.100000E+01	0.242560E+05
172.800	1.000	207.36	0.833333E+00	0.202134E+05
172.800	1.000	248.83	0.694444E+00	0.168445E+05
172.800	1.000	298.60	0.578704E+00	0.140371E+05
172.800	1.000	358.32	0.482253E+00	0.116976E+05
172.800	1.000	429.98	0.401878E+00	0.974796E+04
172.800	1.000	515.98	0.334898E+00	0.812330E+04
172.800	1.000	619.17	0.279082E+00	0.676942E+04
172.800	1.000	743.01	0.232568E+00	0.564118E+04
172.800	1.000	891.61	0.193807E+00	0.470098E+04

R/H = 0.207360E+03

207.360	1.000	207.36	0.100000E+01	0.349287E+05
207.360	1.000	248.83	0.833333E+00	0.291073E+05
207.360	1.000	298.60	0.694444E+00	0.242560E+05
207.360	1.000	358.32	0.578704E+00	0.202134E+05
207.360	1.000	429.98	0.482253E+00	0.168445E+05
207.360	1.000	515.98	0.401878E+00	0.140371E+05
207.360	1.000	619.17	0.334898E+00	0.116976E+05
207.360	1.000	743.01	0.279082E+00	0.974796E+04
207.360	1.000	891.61	0.232568E+00	0.812330E+04
207.360	1.000	1069.93	0.193807E+00	0.676942E+04

TABLE 13. PARAMETER EWH/P FOR VARIOUS R/H AND R/L RATIOS FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A CENTRALLY LOCATED INWARD POINT LOAD BY FOUR ANALYTICAL METHODS

E*W*H/P

R/H = 0.100000E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
100.000	1.000	100.00	0.100000E+01	0.115627E+03	0.111595E+03	0.110917E+03	0.112321E+03
100.000	1.000	120.00	0.833333E+00	0.126663E+03	0.123156E+03	0.122459E+03	0.123925E+03
100.000	1.000	144.00	0.694444E+00	0.138752E+03	0.135860E+03	0.135141E+03	0.136630E+03
100.000	1.000	172.80	0.578704E+00	0.151995E+03	0.149839E+03	0.149099E+03	0.150369E+03
100.000	1.000	207.36	0.482253E+00	0.166502E+03	0.165243E+03	0.164479E+03	0.165543E+03
100.000	1.000	248.83	0.401878E+00	0.182394E+03	0.182243E+03	0.181450E+03	0.183277E+03
100.000	1.000	298.60	0.334898E+00	0.199803E+03	0.201059E+03	0.200222E+03	0.204024E+03
100.000	1.000	358.32	0.279082E+00	0.218873E+03	0.221879E+03	0.220976E+03	0.225798E+03
100.000	1.000	429.98	0.232568E+00	0.239763E+03	0.244962E+03	0.243952E+03	0.245409E+03
100.000	1.000	515.98	0.193807E+00	0.262648E+03	0.270975E+03	0.269789E+03	0.262793E+03

R/H = 0.120000E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
120.000	1.000	120.00	0.100000E+01	0.145223E+03	0.139757E+03	0.139047E+03	0.141494E+03
120.000	1.000	144.00	0.833333E+00	0.159083E+03	0.154217E+03	0.153485E+03	0.155985E+03
120.000	1.000	172.80	0.694444E+00	0.174267E+03	0.170101E+03	0.169347E+03	0.171955E+03
120.000	1.000	207.36	0.578704E+00	0.190900E+03	0.187567E+03	0.186793E+03	0.189312E+03
120.000	1.000	248.83	0.482253E+00	0.209121E+03	0.206802E+03	0.206003E+03	0.208082E+03
120.000	1.000	298.60	0.401878E+00	0.229080E+03	0.228007E+03	0.227179E+03	0.229385E+03
120.000	1.000	358.32	0.334898E+00	0.250945E+03	0.251434E+03	0.250561E+03	0.254767E+03
120.000	1.000	429.98	0.279082E+00	0.274896E+03	0.277367E+03	0.276425E+03	0.283416E+03
120.000	1.000	515.98	0.232568E+00	0.301134E+03	0.306011E+03	0.304960E+03	0.311190E+03
120.000	1.000	619.17	0.193807E+00	0.329875E+03	0.337954E+03	0.336727E+03	0.334995E+03

R/H = 0.144000E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
144.000	1.000	144.00	0.100000E+01	0.182394E+03	0.174922E+03	0.174196E+03	0.178200E+03
144.000	1.000	172.80	0.833333E+00	0.199803E+03	0.193015E+03	0.192265E+03	0.196292E+03
144.000	1.000	207.36	0.694444E+00	0.218873E+03	0.212881E+03	0.212109E+03	0.216262E+03
144.000	1.000	248.83	0.578704E+00	0.239763E+03	0.234716E+03	0.233923E+03	0.238195E+03
144.000	1.000	298.60	0.482253E+00	0.262648E+03	0.253742E+03	0.257925E+03	0.261816E+03
144.000	1.000	358.32	0.401878E+00	0.287716E+03	0.285211E+03	0.284363E+03	0.287726E+03
144.000	1.000	429.98	0.334898E+00	0.315177E+03	0.314401E+03	0.313507E+03	0.318148E+03
144.000	1.000	515.98	0.279082E+00	0.345259E+03	0.346674E+03	0.345709E+03	0.354205E+03
144.000	1.000	619.17	0.232568E+00	0.378213E+03	0.382328E+03	0.381251E+03	0.392322E+03
144.000	1.000	743.01	0.193807E+00	0.414311E+03	0.421721E+03	0.420465E+03	0.426385E+03

R/H = 0.172800E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
172.800	1.000	172.80	0.100000E+01	0.229080E+03	0.218787E+03	0.218071E+03	0.224362E+03
172.800	1.000	207.36	0.833333E+00	0.250945E+03	0.241433E+03	0.240692E+03	0.246995E+03
172.800	1.000	248.83	0.694444E+00	0.274896E+03	0.266288E+03	0.265526E+03	0.271900E+03
172.800	1.000	298.60	0.578704E+00	0.301134E+03	0.293593E+03	0.292810E+03	0.299445E+03
172.800	1.000	358.32	0.482253E+00	0.329875E+03	0.323618E+03	0.322810E+03	0.329406E+03
172.800	1.000	429.98	0.401878E+00	0.361360E+03	0.356667E+03	0.355827E+03	0.361597E+03
172.800	1.000	515.98	0.334898E+00	0.395851E+03	0.393072E+03	0.392185E+03	0.398034E+03
172.800	1.000	619.17	0.279082E+00	0.433633E+03	0.433229E+03	0.432269E+03	0.441833E+03
172.800	1.000	743.01	0.232568E+00	0.475021E+03	0.477602E+03	0.476526E+03	0.491784E+03
172.800	1.000	891.61	0.193807E+00	0.520359E+03	0.526457E+03	0.525198E+03	0.540260E+03

R/H = 0.207360E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
207.360	1.000	207.36	0.100000E+01	0.287716E+03	0.273444E+03	0.272779E+03	0.282393E+03
207.360	1.000	248.83	0.833333E+00	0.315177E+03	0.301794E+03	0.301106E+03	0.310754E+03
207.360	1.000	298.60	0.694444E+00	0.345259E+03	0.332900E+03	0.332191E+03	0.341054E+03
207.360	1.000	358.32	0.578704E+00	0.378213E+03	0.367057E+03	0.366327E+03	0.376235E+03
207.360	1.000	429.98	0.482253E+00	0.414311E+03	0.404592E+03	0.403837E+03	0.414122E+03
207.360	1.000	515.98	0.401878E+00	0.453855E+03	0.445870E+03	0.445081E+03	0.454830E+03
207.360	1.000	619.17	0.334898E+00	0.497173E+03	0.491299E+03	0.490459E+03	0.499209E+03
207.360	1.000	743.01	0.279082E+00	0.544626E+03	0.541306E+03	0.540392E+03	0.551448E+03
207.360	1.000	891.61	0.232568E+00	0.596608E+03	0.596464E+03	0.595430E+03	0.613966E+03
207.360	1.000	1069.93	0.193807E+00	0.653551E+03	0.657216E+03	0.655993E+03	0.680457E+03

TABLE 14. PARAMETER EWH/P FOR VARIOUS R/H AND R/L RATIOS FOR A PINCHED CYLINDRICAL SHELL BY TIMOSHENKO'S INEXTENSIONAL SOLUTION

E*H*H/P

R/H = 0.100000E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
100.000	1.000	200.00	0.500000E+00	0.406165E+04
100.000	1.000	240.00	0.416667E+00	0.338471E+04
100.000	1.000	288.00	0.347222E+00	0.282059E+04
100.000	1.000	345.60	0.289352E+00	0.235049E+04
100.000	1.000	414.72	0.241127E+00	0.195874E+04
100.000	1.000	497.66	0.200939E+00	0.163229E+04
100.000	1.000	597.20	0.167449E+00	0.136024E+04
100.000	1.000	716.64	0.139541E+00	0.113353E+04
100.000	1.000	859.96	0.116284E+00	0.944610E+03
100.000	1.000	1031.96	0.969033E-01	0.787175E+03

R/H = 0.120000E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
120.000	1.000	240.00	0.500000E+00	0.584878E+04
120.000	1.000	288.00	0.416667E+00	0.487398E+04
120.000	1.000	345.60	0.347222E+00	0.406165E+04
120.000	1.000	414.72	0.289352E+00	0.338471E+04
120.000	1.000	497.66	0.241127E+00	0.282059E+04
120.000	1.000	597.20	0.200939E+00	0.235049E+04
120.000	1.000	716.64	0.167449E+00	0.195874E+04
120.000	1.000	859.96	0.139541E+00	0.163229E+04
120.000	1.000	1031.96	0.116284E+00	0.136024E+04
120.000	1.000	1238.35	0.969033E-01	0.113353E+04

R/H = 0.144000E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
144.000	1.000	288.00	0.500000E+00	0.842224E+04
144.000	1.000	345.60	0.[illegible]6667E+00	0.701853E+04
144.000	1.000	414.72	0.347222E+00	0.584878E+04
144.000	1.000	497.66	0.289352E+00	0.487398E+04
144.000	1.000	597.20	0.241127E+00	0.406165E+04
144.000	1.000	716.64	0.200939E+00	0.338471E+04
144.000	1.000	859.96	0.167449E+00	0.282059E+04
144.000	1.000	1031.96	0.139541E+00	0.235049E+04
144.000	1.000	1238.35	0.116284E+00	0.195874E+04
144.000	1.000	1486.02	0.969033E-01	0.163229E+04

R/H = 0.172800E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
172.800	1.000	345.60	0.500000E+00	0.121280E+05
172.800	1.000	414.72	0.416667E+00	0.101067E+05
172.800	1.000	497.66	0.347222E+00	0.842224E+04
172.800	1.000	597.20	0.289352E+00	0.701853E+04
172.800	1.000	716.64	0.241127E+00	0.584878E+04
172.800	1.000	859.96	0.200939E+00	0.487398E+04
172.800	1.000	1031.96	0.167449E+00	0.406165E+04
172.800	1.000	1238.35	0.139541E+00	0.338471E+04
172.800	1.000	1486.02	0.116284E+00	0.282059E+04
172.800	1.000	1783.22	0.969033E-01	0.235049E+04

R/H = 0.207360E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
207.360	1.000	414.72	0.500000E+00	0.174644E+05
207.360	1.000	497.66	0.416667E+00	0.145536E+05
207.360	1.000	597.20	0.347222E+00	0.121280E+05
207.360	1.000	716.64	0.289352E+00	0.101067E+05
207.360	1.000	859.96	0.241127E+00	0.842224E+04
207.360	1.000	1031.96	0.200939E+00	0.701853E+04
207.360	1.000	1238.35	0.167449E+00	0.584878E+04
207.360	1.000	1486.02	0.139541E+00	0.487398E+04
207.360	1.000	1783.22	0.116284E+00	0.406165E+04
207.360	1.000	2139.86	0.969033E-01	0.338471E+04

TABLE 15. PARAMETER EWH/P FOR VARIOUS R/H AND R/L RATIOS FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A CENTRALLY LOCATED INWARD POINT LOAD BY FOUR ANALYTICAL METHODS

E*W*H/P

R/H = 0.100000E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
100.000	1.000	200.00	0.500000E+00	0.163521E+03	0.162068E+03	0.161310E+03	0.162372E+03
100.000	1.000	240.00	0.416667E+00	0.179128E+03	0.178737E+03	0.177950E+03	0.179509E+03
100.000	1.000	288.00	0.347222E+00	0.196225E+03	0.197175E+03	0.196348E+03	0.199729E+03
100.000	1.000	345.60	0.289352E+00	0.214954E+03	0.217586E+03	0.216698E+03	0.221568E+03
100.000	1.000	414.72	0.241127E+00	0.235470E+03	0.240179E+03	0.239194E+03	0.241774E+03
100.000	1.000	497.66	0.200939E+00	0.257944E+03	0.265556E+03	0.264413E+03	0.259341E+03
100.000	1.000	597.20	0.167449E+00	0.282564E+03	0.294137E+03	0.292725E+03	0.278556E+03
100.000	1.000	716.64	0.139541E+00	0.309533E+03	0.325179E+03	0.323350E+03	0.307786E+03
100.000	1.000	859.96	0.116284E+00	0.339077E+03	0.359162E+03	0.356709E+03	0.356590E+03
100.000	1.000	1031.96	0.969033E-01	0.371440E+03	0.400743E+03	0.397116E+03	0.432574E+03

R/H = 0.120000E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
120.000	1.000	240.00	0.500000E+00	0.205376E+03	0.202839E+03	0.202046E+03	0.204215E+03
120.000	1.000	288.00	0.416667E+00	0.224978E+03	0.223637E+03	0.222815E+03	0.224874E+03
120.000	1.000	345.60	0.347222E+00	0.246451E+03	0.246600E+03	0.245736E+03	0.249395E+03
120.000	1.000	414.72	0.289352E+00	0.269974E+03	0.272018E+03	0.271092E+03	0.277648E+03
120.000	1.000	497.66	0.241127E+00	0.295741E+03	0.300101E+03	0.299076E+03	0.305972E+03
120.000	1.000	597.20	0.200939E+00	0.323968E+03	0.331309E+03	0.330125E+03	0.330539E+03
120.000	1.000	716.64	0.167449E+00	0.354889E+03	0.366563E+03	0.365108E+03	0.353293E+03
120.000	1.000	859.96	0.139541E+00	0.388762E+03	0.405602E+03	0.403703E+03	0.383149E+03
120.000	1.000	1031.96	0.116284E+00	0.425867E+03	0.447471E+03	0.444917E+03	0.432559E+03
120.000	1.000	1238.35	0.969033E-01	0.466514E+03	0.495447E+03	0.491830E+03	0.513589E+03

R/H = 0.144000E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
144.000	1.000	288.00	0.500000E+00	0.257944E+03	0.253794E+03	0.252982E+03	0.256998E+03
144.000	1.000	345.60	0.416667E+00	0.282564E+03	0.279757E+03	0.278917E+03	0.282314E+03
144.000	1.000	414.72	0.347222E+00	0.309533E+03	0.308382E+03	0.307499E+03	0.311653E+03
144.000	1.000	497.66	0.289352E+00	0.339077E+03	0.340014E+03	0.339066E+03	0.346705E+03
144.000	1.000	597.20	0.241127E+00	0.371440E+03	0.374984E+03	0.373933E+03	0.384919E+03
144.000	1.000	716.64	0.200939E+00	0.406892E+03	0.413570E+03	0.412357E+03	0.420121E+03
144.000	1.000	859.96	0.167449E+00	0.445728E+03	0.456782E+03	0.455302E+03	0.450093E+03
144.000	1.000	1031.96	0.139541E+00	0.488270E+03	0.505387E+03	0.503451E+03	0.482362E+03
144.000	1.000	1238.35	0.116284E+00	0.534873E+03	0.557971E+03	0.555328E+03	0.531868E+03
144.000	1.000	1486.02	0.969033E-01	0.585924E+03	0.615218E+03	0.611528E+03	0.615579E+03

R/H = 0.172800E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
172.800	1.000	345.60	0.500000E+00	0.323968E+03	0.317436E+03	0.316634E+03	0.323303E+03
172.800	1.000	414.72	0.416667E+00	0.354889E+03	0.349860E+03	0.349027E+03	0.354983E+03
172.800	1.000	497.66	0.347222E+00	0.388762E+03	0.385573E+03	0.384697E+03	0.390303E+03
172.800	1.000	597.20	0.289352E+00	0.425867E+03	0.424947E+03	0.424004E+03	0.432519E+03
172.800	1.000	716.64	0.241127E+00	0.466514E+03	0.468458E+03	0.467409E+03	0.481702E+03
172.800	1.000	859.96	0.200939E+00	0.511041E+03	0.516395E+03	0.515179E+03	0.531183E+03
172.800	1.000	1031.96	0.167449E+00	0.559817E+03	0.569445E+03	0.567961E+03	0.573638E+03
172.800	1.000	1238.35	0.139541E+00	0.613249E+03	0.629236E+03	0.627298E+03	0.612188E+03
172.800	1.000	1486.02	0.116284E+00	0.671781E+03	0.695350E+03	0.692662E+03	0.662631E+03
172.800	1.000	1783.22	0.969033E-01	0.735899E+03	0.766045E+03	0.762249E+03	0.747050E+03

R/H = 0.207360E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
207.360	1.000	414.72	0.500000E+00	0.406892E+03	0.396867E+03	0.396117E+03	0.406363E+03
207.360	1.000	497.66	0.416667E+00	0.445728E+03	0.437372E+03	0.436590E+03	0.446549E+03
207.360	1.000	597.20	0.347222E+00	0.488270E+03	0.481944E+03	0.481117E+03	0.489949E+03
207.360	1.000	716.64	0.289352E+00	0.534873E+03	0.531005E+03	0.530109E+03	0.540257E+03
207.360	1.000	859.96	0.241127E+00	0.585924E+03	0.585093E+03	0.584087E+03	0.600915E+03
207.360	1.000	1031.96	0.200939E+00	0.641848E+03	0.644722E+03	0.643543E+03	0.667542E+03
207.360	1.000	1238.35	0.167449E+00	0.703109E+03	0.710239E+03	0.708788E+03	0.728776E+03
207.360	1.000	1486.02	0.139541E+00	0.770218E+03	0.783360E+03	0.781463E+03	0.780153E+03
207.360	1.000	1783.22	0.116284E+00	0.843731E+03	0.865603E+03	0.862937E+03	0.834782E+03
207.360	1.000	2139.86	0.969033E-01	0.924261E+03	0.954664E+03	0.950790E+03	0.919082E+03

TABLE 16. PARAMETER EWH/P FOR VARIOUS R/H AND R/L RATIOS FOR A PINCHED CYLINDRICAL SHELL BY TIMOSHENKO'S INEXTENSIONAL SOLUTION

E*W*H/P

R/H = 0.200000E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
200.000	1.000	200.00	0.100000E+01	0.324932E+05
200.000	1.000	240.00	0.833333E+00	0.270777E+05
200.000	1.000	288.00	0.694444E+00	0.225647E+05
200.000	1.000	345.60	0.578704E+00	0.188039E+05
200.000	1.000	414.72	0.482253E+00	0.156699E+05
200.000	1.000	497.66	0.401878E+00	0.130583E+05
200.000	1.000	597.20	0.334898E+00	0.108819E+05
200.000	1.000	716.64	0.279082E+00	0.906826E+04
200.000	1.000	859.96	0.232568E+00	0.755688E+04
200.000	1.000	1031.96	0.193807E+00	0.629740E+04

R/H = 0.240000E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
240.000	1.000	240.00	0.100000E+01	0.467902E+05
240.000	1.000	288.00	0.833333E+00	0.389918E+05
240.000	1.000	345.60	0.694444E+00	0.324932E+05
240.000	1.000	414.72	0.578704E+00	0.270777E+05
240.000	1.000	497.66	0.482253E+00	0.225647E+05
240.000	1.000	597.20	0.401878E+00	0.188039E+05
240.000	1.000	716.64	0.334898E+00	0.156699E+05
240.000	1.000	859.96	0.279082E+00	0.130583E+05
240.000	1.000	1031.96	0.232568E+00	0.108819E+05
240.000	1.000	1238.35	0.193807E+00	0.906826E+04

R/H = 0.288000E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
288.000	1.000	288.00	0.100000E+01	0.673779E+05
288.000	1.000	345.60	0.833333E+00	0.561483E+05
288.000	1.000	414.72	0.694444E+00	0.467902E+05
288.000	1.000	497.66	0.578704E+00	0.389918E+05
288.000	1.000	597.20	0.482253E+00	0.324932E+05
288.000	1.000	716.64	0.401878E+00	0.270777E+05
288.000	1.000	859.96	0.334898E+00	0.225647E+05
288.000	1.000	1031.96	0.279082E+00	0.188039E+05
288.000	1.000	1238.35	0.232568E+00	0.156699E+05
288.000	1.000	1486.02	0.193807E+00	0.130583E+05

R/H = 0.345600E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
345.600	1.000	345.60	0.100000E+01	0.970242E+05
345.600	1.000	414.72	0.833333E+00	0.808535E+05
345.600	1.000	497.66	0.694444E+00	0.673779E+05
345.600	1.000	597.20	0.578704E+00	0.561483E+05
345.600	1.000	716.64	0.482253E+00	0.467902E+05
345.600	1.000	859.96	0.401878E+00	0.389918E+05
345.600	1.000	1031.96	0.334898E+00	0.324932E+05
345.600	1.000	1238.35	0.279082E+00	0.270777E+05
345.600	1.000	1486.02	0.232568E+00	0.225647E+05
345.600	1.000	1783.22	0.193807E+00	0.188039E+05

R/H = 0.414720E+03

RADIUS	THICKNESS	LENGTH	R/L	INEXTENSIONAL
414.720	1.000	414.72	0.100000E+01	0.139715E+06
414.720	1.000	497.66	0.833333E+00	0.116429E+06
414.720	1.000	597.20	0.694444E+00	0.970242E+05
414.720	1.000	716.64	0.578704E+00	0.808535E+05
414.720	1.000	859.96	0.482253E+00	0.673779E+05
414.720	1.000	1031.96	0.401878E+00	0.561483E+05
414.720	1.000	1238.35	0.334898E+00	0.467902E+05
414.720	1.000	1486.02	0.279082E+00	0.389918E+05
414.720	1.000	1783.22	0.232568E+00	0.324932E+05
414.720	1.000	2139.86	0.193807E+00	0.270777E+05

TABLE 17. PARAMETER EWH/P FOR VARIOUS R/H AND R/L RATIOS FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL SUBJECT TO A CENTRALLY LOCATED INWARD POINT LOAD BY FOUR ANALYTICAL METHODS

E*H*H/P

R/H = 0.200000E+03

RADIUS	THICKNESS	LENGTH	R/L	ODQVIST	BIJLAARD	MIZOGUCHI	YUAN-TING
200.000	1.000	200.00	0.100000E+01	0.275008E+03	0.261638E+03	0.260958E+03	0.269812E+03
200.000	1.000	240.00	0.833333E+00	0.301256E+03	0.288752E+03	0.288050E+03	0.296931E+03
200.000	1.000	288.00	0.694444E+00	0.330010E+03	0.318505E+03	0.317781E+03	0.326685E+03
200.000	1.000	345.60	0.578704E+00	0.361507E+03	0.351179E+03	0.350433E+03	0.359598E+03
200.000	1.000	414.72	0.482253E+00	0.396012E+03	0.387083E+03	0.386318E+03	0.395784E+03
200.000	1.000	497.66	0.401878E+00	0.433809E+03	0.426535E+03	0.425782E+03	0.434608E+03
200.000	1.000	597.20	0.334898E+00	0.475214E+03	0.470061E+03	0.469209E+03	0.477206E+03
200.000	1.000	716.64	0.279082E+00	0.520571E+03	0.517937E+03	0.517010E+03	0.527674E+03
200.000	1.000	859.96	0.232568E+00	0.570257E+03	0.570770E+03	0.569724E+03	0.587668E+03
200.000	1.000	1031.96	0.193807E+00	0.624685E+03	0.628952E+03	0.627718E+03	0.650356E+03

R/H = 0.240000E+03

240.000	1.000	240.00	0.100000E+01	0.345400E+03	0.326771E+03	0.326194E+03	0.339521E+03
240.000	1.000	288.00	0.833333E+00	0.378366E+03	0.360724E+03	0.360124E+03	0.373515E+03
240.000	1.000	345.60	0.694444E+00	0.414480E+03	0.397968E+03	0.397346E+03	0.410754E+03
240.000	1.000	414.72	0.578704E+00	0.454040E+03	0.438849E+03	0.438206E+03	0.451765E+03
240.000	1.000	497.66	0.482253E+00	0.497375E+03	0.483752E+03	0.483083E+03	0.497226E+03
240.000	1.000	597.20	0.401878E+00	0.544848E+03	0.533098E+03	0.532395E+03	0.546592E+03
240.000	1.000	716.64	0.334898E+00	0.596851E+03	0.587357E+03	0.586603E+03	0.599480E+03
240.000	1.000	859.96	0.279082E+00	0.653817E+03	0.647022E+03	0.646189E+03	0.659703E+03
240.000	1.000	1031.96	0.232568E+00	0.716221E+03	0.712679E+03	0.711723E+03	0.732619E+03
240.000	1.000	1238.35	0.193807E+00	0.784580E+03	0.785005E+03	0.783855E+03	0.815315E+03

R/H = 0.288000E+03

288.000	1.000	288.00	0.100000E+01	0.433809E+03	0.407722E+03	0.407336E+03	0.427143E+03
288.000	1.000	345.60	0.833333E+00	0.475214E+03	0.450246E+03	0.449837E+03	0.469734E+03
288.000	1.000	414.72	0.694444E+00	0.520571E+03	0.496873E+03	0.496449E+03	0.516432E+03
288.000	1.000	497.66	0.578704E+00	0.570257E+03	0.548043E+03	0.547591E+03	0.567636E+03
288.000	1.000	597.20	0.482253E+00	0.624685E+03	0.604206E+03	0.603728E+03	0.624344E+03
288.000	1.000	716.64	0.401878E+00	0.684308E+03	0.665877E+03	0.665362E+03	0.686876E+03
288.000	1.000	859.96	0.334898E+00	0.749622E+03	0.733607E+03	0.733038E+03	0.753832E+03
288.000	1.000	1031.96	0.279082E+00	0.821169E+03	0.807993E+03	0.807342E+03	0.826941E+03
288.000	1.000	1238.35	0.232568E+00	0.899546E+03	0.889648E+03	0.888869E+03	0.913756E+03
288.000	1.000	1486.02	0.193807E+00	0.985403E+03	0.979429E+03	0.978452E+03	0.101772E+04

R/H = 0.345600E+03

345.600	1.000	345.60	0.100000E+01	0.544848E+03	0.508159E+03	0.508098E+03	0.537260E+03
345.600	1.000	414.72	0.833333E+00	0.596851E+03	0.561429E+03	0.561345E+03	0.590627E+03
345.600	1.000	497.66	0.694444E+00	0.653817E+03	0.619829E+03	0.619723E+03	0.649183E+03
345.600	1.000	597.20	0.578704E+00	0.716221E+03	0.683878E+03	0.683749E+03	0.713331E+03
345.600	1.000	716.64	0.482253E+00	0.784580E+03	0.754141E+03	0.753984E+03	0.783952E+03
345.600	1.000	859.96	0.401878E+00	0.859465E+03	0.831232E+03	0.831036E+03	0.862417E+03
345.600	1.000	1031.96	0.334898E+00	0.941497E+03	0.915800E+03	0.915547E+03	0.947728E+03
345.600	1.000	1238.35	0.279082E+00	0.103136E+04	0.100855E+04	0.100821E+04	0.103877E+04
345.600	1.000	1486.02	0.232568E+00	0.112980E+04	0.111019E+04	0.110971E+04	0.114222E+04
345.600	1.000	1783.22	0.193807E+00	0.123763E+04	0.122161E+04	0.122093E+04	0.126810E+04

R/H = 0.414720E+03

414.720	1.000	414.72	0.100000E+01	0.684308E+03	0.632532E+03	0.632968E+03	0.675695E+03
414.720	1.000	497.66	0.833333E+00	0.749622E+03	0.699277E+03	0.699709E+03	0.742538E+03
414.720	1.000	597.20	0.694444E+00	0.821169E+03	0.772427E+03	0.772836E+03	0.815903E+03
414.720	1.000	716.64	0.578704E+00	0.899546E+03	0.852618E+03	0.853004E+03	0.896398E+03
414.720	1.000	859.96	0.482253E+00	0.985403E+03	0.940537E+03	0.940894E+03	0.984615E+03
414.720	1.000	1031.96	0.401878E+00	0.107946E+04	0.103692E+04	0.103723E+04	0.108238E+04
414.720	1.000	1238.35	0.334898E+00	0.118248E+04	0.114254E+04	0.114279E+04	0.119044E+04
414.720	1.000	1486.02	0.279082E+00	0.129535E+04	0.125819E+04	0.125835E+04	0.130601E+04
414.720	1.000	1783.22	0.232568E+00	0.141898E+04	0.138476E+04	0.138477E+04	0.143173E+04
414.720	1.000	2139.86	0.193807E+00	0.155442E+04	0.152319E+04	0.152299E+04	0.153123E+04

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NOMENCLATURE

C_m, C_m^k = Fourier coefficient

D = Flexural rigidity (= $\frac{Eh^3}{12(1-\nu^2)}$)

E = Young's Modulus

h,H* = Thickness of shell

J = $\sqrt{-1}$

L, ℓ = Length of shell

M_x = Bending moment/unit length of a section perpendicular to the axis of the cylindrical shell, i.e., per unit of circumference

$M_{x\varphi}$ = Torsional moment/unit length of an axial section of cylindrical shell

M_φ = Bending moment/unit length of axial section

m = Axial harmonic index

n = Peripheral harmonic index

P = Concentrated force applied radially

$P_{m,n}$ = Fourier coefficient

R = Mean shell radius

RU,RV,RW = Rotations about local (U,V,W) axes (Figure 1)

s = Peripheral coordinate along a circular arc on cylindrical shell (= $R\varphi$)

x = Axial coordinate (Figure 1)

(U,V,W) = Displacements along local axial, local tangential, and local radial axes (Figure 1)

*Except in Equation (12)

NOMENCLATURE (Cont.)

W	= Radial displacement under point load
w	= Arbitrary radial displacement
$Z_{m,n}$	= Fourier coefficient
α_m	= Phase angle for m_{th} Fourier component
ζ	= Axial coordinate x normalized with respect to length of cylinder L
λ	= Parameter in expansion
$\lambda_m^{(k)}$	= Root of characteristic equation
θ, φ	= Angle indicating portion of cylindrical shell analyzed
ν	= Poisson's ratio
$\rho_I(x,y)$	= Particular integral of 8th order partial differential equation in radial displacement
$\rho_{II}(x,y)$	= Homogeneous solution of partial differential equation
$\rho_m(x)$	= Coefficient in expansion
$\varphi_m(\zeta)$	= Coefficient of expansion as a function of axial coordinate
$\varphi_n(\varphi)$	= Coefficient of expansion as a function of circumferential coordinate

APPENDIX A

SOLUTION METHODS

In this Appendix, we focus our attention to closed form solutions, which are primarily based on linear shell theories,[A-1 - A-10] for the response of cylindrical shells subjected to static point loads. Small displacements, strains, rotations, and linear elastic material are assumed throughout this work. Solutions based on three-dimensional elasticity theory are substantially more complex and will only be mentioned.

The linear shell equations can be written as a partial differential equation of eighth order in the radial displacement w by eliminating the other two deformation measures, u and v. Such are the full Fluegge,[A-1] simplified Donnell,[A-11] improved Donnell,[A-12] Morley,[A-13, A-14] etc., shell equations. If certain terms are neglected, the range of applicability can be impaired. In Appendix C it is demonstrated how Timoshenko equations[A-2] lead to a nonsymmetric coefficient matrix in contrast to those of Vlasov[A-15] or Fluegge.[A-1]

Most of the solutions obtained in the literature are based on the simplified Donnell[A-11, A-16] equation, rather than the complete Donnell,[A-12] the full Fluegge,[A-1] Morley,[A-13, A-14] Schorer,[A-17, A-18] etc., equations. There are numerous solutions based on Novozhilov,[A-9] Sanders,[A-10] and other theories critically analyzed in Buchwald.[A-19,A-20] There[A-19, A-20] it is shown, through asymptotic expansions that different solutions in the radial displacement w are required for an "inner" and "outer" regions. Some of the simplified equations such as Donnell's[A-11, A-16] or Morley's[A-13, A-14] attempt to cover the entire range. They contain incorrect second order terms in both regions, and, therefore, in limiting cases the dominating behavior may not be correct. In each region the second order terms must not be included in the approximate theories for limiting behavior, although the first order solution over the entire cylinder may be correct.

Before reviewing the pertinent literature, a few words are needed to establish the general framework of solution methods presently available to cylindrical shells (Figure 1 of the main text) subject to a variety of loads and end boundary conditions. The theory of variational or energy based techniques (such as finite element method (FEM))[A-21] will not be referred to here.

The analytical methods based on shell theory can be grouped in the following classes, irrespective of the partial differential equation, which must be solved.

SEPARATION OF VARIABLES

Essentially most of the shell solutions are of this form, except for the ones by asymptotic methods. The method of solution may proceed under either 1, 2, 3, or 4 below:

1. Resolve the loading function into a double Fourier series of the form

$$\sum_m \sum_n P_{mn} \sin(m\pi x/L) \sin(n\varphi)$$

for simply supported ends.[A-22, A-23] The choice of trigonometric products must be such that the boundary conditions are satisfied automatically. This method was originally given by H. Reissner[A-22] for a simply supported cylinder. To accelerate convergence one may resort to tricks such as subtracting and adding a series corresponding to a reference plate problem.[A-2] The solution of the plate problem is obtained by other techniques. The direct approach is to incorporate a large number of terms in the series. If this last route is followed, calculation of stress resultants and couples becomes less accurate than displacements due to the term by term successive differentiations.

2. Resolve the loading function and displacements in a single Fourier series peripherally and exponential functions axially of the form

$$\sum_m C_m \exp(\lambda\zeta) \sin(m\varphi).$$

The representation by a Fourier series peripherally and Fourier integral axially is grouped under this class too. It is worth noting that slowly damped edge disturbances must be included to complete the solution. This method is suitable to treat line loads on generatrices or loads applied at edges x=const.

3. Resolve the radial displacement w in a single Fourier series axially and damped trigonometric function peripherally of the form

$$\sum_n \phi_n (\varphi) \sin\left(\frac{n\pi x}{L}\right) \text{ or } \sum_n C_n \exp(n\varphi) \sin(n\varphi) \sin\left(\frac{n\pi x}{L}\right)$$

Edge disturbances from the generatrix are needed to complete the solution. This method is better suited to treating loads at the edges φ=const.

4. Obtain the particular integral, $\rho_I(x,\varphi)$, using 1 above. Resolve the solution of the homogeneous 8th order equation in a series of the form

$$\rho_{II}(x,\varphi) = \sum_{m=0,1}^{\infty} \bar{\rho}_m(x) \cos[m(\varphi + \alpha_m)]$$

where $\bar{\rho}_m(x)$ is the solution of the 8th order linear homogenous ordinary differential equation with constant coefficients and satisfies the boundary conditions.[A-24] Similar stress function methods can be found in References A-15, A-25, and A-26.

As an additional comment, we note that, if a solution to Donnell's shell equations is known, then we may obtain solutions of the full Fluegge equation by using 1, 2, 3, or 4. Correction functions[A-27] are obtained by the previous method of separation of variables.

TRANSFORM METHODS (A form of Separation of Variables)

If the equations of elasticity are employed, then the infinitely long shell can be treated by the following:

a. Fourier integral axially, Fourier series peripherally, and stress functions methods,[A-28]

b. Fourier integral transform methods with associated stress function.[A-29]

ASYMPTOTIC INTEGRATION METHODS

These techniques can be found in works by E. Reissner[A-30] and Koiter[A-31] and references listed therein.

At this stage it must be mentioned that the eighth order algebraic characteristic equation has eight complex roots. Of these, two are the so-called principal roots. They are such that their real parts are negative and imaginary parts are positive (second quadrant on complex plane). These are the highly and lightly damped roots. However, for the two particular values of m (m=0 or m=1) the characteristic equation has zero roots of multiplicity 4. The assumed form

$$w(\zeta,\varphi) = \sum_{m=0,1}^{\infty} \phi_m(\zeta)\cos(m\varphi) = \sum_{m=0,1}^{\infty}\sum_{k=1}^{k=8} C_m^k \exp(\lambda_m^{(k)}\zeta)\cos(m\varphi)$$

does not account for all of the zero roots of multiplicity 4. There are 12 elementary states of stress that correspond to the zero roots.[A-6] Six are rigid body motions, while the other six are elementary states of stress with the shell looked upon as a beam.

HISTORICAL REVIEW

Linear shell theories have been developed by Novozhilov,[A-9] Gol'denveizer,[A-6] E. Reissner,[A-30, A-32] Fluegge,[A-1] Love,[A-7] Timoshenko,[A-2] Sanders Jr.,[A-10] Koiter,[A-31, A-33] and Vlasov.[A-15, A-25]

The earlier work on specific problems on circular cylindrical shells was carried out by H. Reissner,[A-22] Miesel,[A-34] Fluegge,[A-1], Aas-Jakobsen,[A-35, A-36] Finsterwalder,[A-37] Schorer,[A-17] Odqvist,[A-18] Donnell,[A-11] Bijlaard,[A-23] Yuan,[A-38] Yuan & Ting,[A-24, A-39] Morley,[A-13,A-14] Hoff,[A-40 - A-42] Kempner,[A-43,A-44] Mizoguchi,[A-45] Seide,[A-46] Yao,[A-28] Bieger,[A-47] Berger,[A-48] Melworm, et al.[A-49] and others.

More specifically, Finsterwalder[A-37] and Schorer[A-17] simplified the various equations under restrictive assumptions. Schorer[A-17] reduced the eighth order homogeneous equation to a fourth order one by neglecting lower order derivatives. He based this on the assumption that damping factors exceed unity; therefore, the higher the derivative, the more dominating it becomes. This, of course, is not always the case. He also introduced additional corrections based on Newton's method to improve upon his solution. Odqvist[A-18] used Schorer's theory[A-17] to obtain the only truly explicit solution for a simply supported circularly cylindrical shell subjected to a concentrated load.

Bijlaard[A-23] solved the circularly cylindrical shell problem subject to a variety of loads and simply supported conditions by using a double Fourier series method. The eighth order differential equation is slightly different from Donnell's[A-11] due to the simplifying assumptions. Limitations of the analysis occur for certain combinations of L/R and R/h, as pointed out by Murphy.[A-23]

Mizoguchi,[A-45] employed his own version of the characteristic equation. He solved the simply supported shell problem by a finite Fourier transform method and the method of images (Figure 2 of the main text). Yuan[A-38] used the simplified Donnell equation[A-11] to obtain a solution of the infinitely long cylinder subject to two self-equilibrating point loads. The load was resolved in a product of a Fourier integral axially and a cosine Fourier series peripherally, since the function is even in both longitudinal and circumferential variables (x,s). Ting & Yuan[A-39] employed the complete Donnell equation.[A-12] In their analysis they treated boundary conditions other than simply supported. The infinite length cylinder solution was used as a particular integral. Boundary conditions were satisfied by obtaining the proper solutions to the homogeneous differential equation. Yuan & Ting[A-24] applied Fluegge's[A-1] full equations. Both loading function and radial displacement were resolved in a product of a Fourier integral axially and a Fourier series peripherally. The displacement of the infinitely long cylinder is obtained first. They used the method of images to obtain the correct end boundary conditions of a simply supported cylinder of finite length.

Seide[A-46] analyzed the bending of an infinitely long cylindrical shell subject to point loads or moments using Donnell's equations. Similar work was done by Melworm, et al.[A-49] Bieger[A-47] reduced the three coupled partial differential equations of Fluegge[A-1] for the three deformation measures (u,v,w) to an eighth order equation for an auxiliary function $F(x,\varphi)$ for the case of a concentrated load. He solved the infinitely long cylinder problem by resolving the load in a product of a Fourier series peripherally and a Fourier integral axially.

Cooper[A-50] employed Naghdi's shallow shell equations, including the effect of transverse shear deformation, in order to obtain solutions to the simply supported cylindrical shell of finite length subject to a line load on the generatrix. His results indicated close agreement stresswise with Donnell[A-11] equation solutions.

Biezeno and Koch[A-51] gave particular solutions to a system of three coupled partial differential equations for an assumed load system radially, tangentially, and axially. Boundary conditions must be accounted for separately. This can also be found in Reference A-5.

Berger[A-48] studied the roots of the homogeneous Fluegge[A-1] equation, assuming that the displacement varied as $\exp(-\lambda x/a)\exp(jm\varphi)$. He gave comparisons with the solutions by Miesel[A-34] and Aas-Jakobsen.[A-35]

This brings us to the topic of eigenvalues of the characteristic equation. Morley[A-14] calculated the eigenvalues of the homogeneous differential equation by Fluegge,[A-1] Donnell,[A-11] and his own version. He showed the proximity of his to Fluegge's for various values of the harmonic index n in contrast to Donnell's,[A-11] which deviates more, the thicker the shell and the smaller n (either peripheral or axial harmonic content) is. Furthermore, special methods were used to solve the eighth order characteristic equation by reducing it to a biquadratic. For a general biquadratic there exist general[A-52] and special methods.[A-53]

Morley[A-14] also solved the inhomogeneous problem for his own equation. A similar study for Donnell's equation[A-11] with axial prestress (tensile or compressive) was conducted by Nachbar,[A-54] whose analysis was very sketchy with missing intermediate results. Some can be recovered from Holand.[A-3] Furthermore, there is a series of articles by Hoff[A-41, A-42] with explicit solutions for the roots of the homogeneous Donnell[A-11] differential equation as well as the membrane stresses and displacements. It was concluded in Reference A-42 that under certain conditions Donnell's equations[A-11] can be used and still be accurate in preference to the more complicated Fluegge[A-1] equations. Further discussion appears in Reference A-44. However, the most extensive study seems to be the books by Holand[A-3] and Seide,[A-55] where other approximations to the large and small roots, apart from previous ones, such as Gruber's,[A-56] Miesel's,[A-31] Aas-Jakobsen's,[A-35] Berger's,[A-48] Olsen's,[A-3] etc., are given and plotted in an attempt to show that, at least for some engineering applications, special savings can be obtained if one resorts to them. Other studies to be mentioned are the works of Houghton, et al.[A-57, A-58] and Holand[A-59] and a recent study on the form of thin shell

equations and their eigenvalues by Akeju.[A-60] Furthermore, analysis by Buchwald[A-19, A-20] indicates the importance of obtaining the correct exact eigenvalues for limiting cases.

In view of experimental evidence[A-61] it is not advisable to neglect the longitudinal moment, M_x, and torsional moment, $M_{x\varphi}$. Consequently, Hoff, et al.[A-40] employed Donnell equations[A-11, A-16] to solve the simply supported cylindrical shell problem subject to numerous loading conditions on a generatrix. The radial displacement w was assumed as a Fourier series axially and an exponential peripherally with the real part negative.

Holand,[A-62] employing Donnell's[A-11] theory for a simply supported cylindrical shell, resolved forces and loads in a single Fourier series longitudinally and a damped trigonometric function peripherally. He was principally interested in obtaining influence surfaces for bending moments. Stafford and Rim[A-27] introduced a new approach by a corrector function to the Donnell[A-11] equation solution through an iterative procedure. Buchwald[A-19,A-20] was concerned with the various differential equations derived from different shell theories as applied to thin circular cylinders. There are regions of validity for each of them (Donnell, Fluegge, complete Donnell, Morley, etc.), i.e., inner and outer regions as well as regions of transition.

Kempner, et al.,[A-43] employed Donnell[A-11] equations to obtain tables and curves for simply supported shells subject to numerous loading conditions for a variety of radius/thickness and radius/length ratios.

Yao[A-28] treated the problem of a thick long cylinder subject to two radial self-equilibrating loads by elasticity theory. The load is represented by a Fourier series circumferentially and a Fourier integral axially. A stress function is constructed and appropriate boundary conditions are satisfied (all stresses must vanish at $+\infty$ & $-\infty$) while stresses at inner and outer radii are prescribed. The radial displacement is finally given in terms of modified Bessel functions.

Klosner[A-29] solved the long circular cylindrical shell problem subjected to uniform external circumferential, radial line loads by employing elasticity equations and Fourier integral transform techniques. The inverse transforms were obtained by numerical and asymptotic integration.

Finally the solution of an infinitely long cylinder, reinforced by a single ring and subjected to self-equilibrating point loads eccentrically applied was solved by Allentuch, et al.[A-63], by a Fourier series-Fourier integral method. Their results were supported by test data.[A-64] The unusual point was that their radial deformation was not a maximum under the ring stiffener. It was claimed to be the effect of eccentricity. To this we must add the inextensional solution by Timoshenko[A-2] (sometimes referred to as Rayleigh's solution) of a circularly cylindrical shell subject to point loads. Note that the ends are free and there is no membrane action in the longitudinal direction of the shell. This solution resembles the ring subject to point loads solution.[A-65]

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APPENDIX B

BIJLAARD'S SOLUTION BY DOUBLE FOURIER SERIES

CIRCULAR CYLINDER OF FINITE LENGTH SUBJECT TO A POINT LOAD AT MIDSPAN SIMPLY SUPPORTED AT BOTH ENDS

The following presentation is based on Bijlaard's analysis.[B-1] We employ Timoshenko's equations[B-2] (eq. 303, p. 513) of equilibrium for the three deformation measures (u,v,w), allowing only radial pressure load, p_r, with E, Young's Modulus, ν, Poisson's ratio, R, the cylinder's mean radius, h, the skin thickness (Figure B-1 shows a point load P applied at the cylindrical shell).

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2R^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{(1+\nu)}{2R}\frac{\partial^2 v}{\partial x \partial \varphi} - \frac{\nu}{R}\frac{\partial w}{\partial x} = 0 \tag{B.1}$$

$$\frac{(1+\nu)}{2R}\frac{\partial^2 u}{\partial x \partial \varphi} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 v}{\partial \varphi^2} - \frac{1}{R^2}\frac{\partial w}{\partial \varphi} + \frac{1}{12}\left[\frac{h}{R}\right]^2\left(\frac{\partial^3 w}{\partial x^2 \partial \varphi} + \frac{1}{R^2}\frac{\partial^3 w}{\partial \varphi^3}\right)$$

$$+ \frac{1}{12}\left[\frac{h}{R}\right]^2\left[(1-\nu)\frac{\partial^2 v}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 v}{\partial \varphi^2}\right] = 0 \tag{B.2}$$

$$\nu\frac{\partial u}{\partial x} + \frac{1}{R}\frac{\partial v}{\partial \varphi} - \frac{w}{R} - \frac{1}{12}Rh^2\nabla^4 w - \frac{1}{12}h^2\left(\frac{2-\nu}{R}\frac{\partial^3 v}{\partial x^2 \partial \varphi} + \frac{1}{R^3}\frac{\partial^3 v}{\partial \varphi^3}\right)$$

$$+ \frac{(1-\nu^2)}{Eh}Rp_r = 0 \tag{B.3}$$

with the abbreviation

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2} \tag{B.4}$$

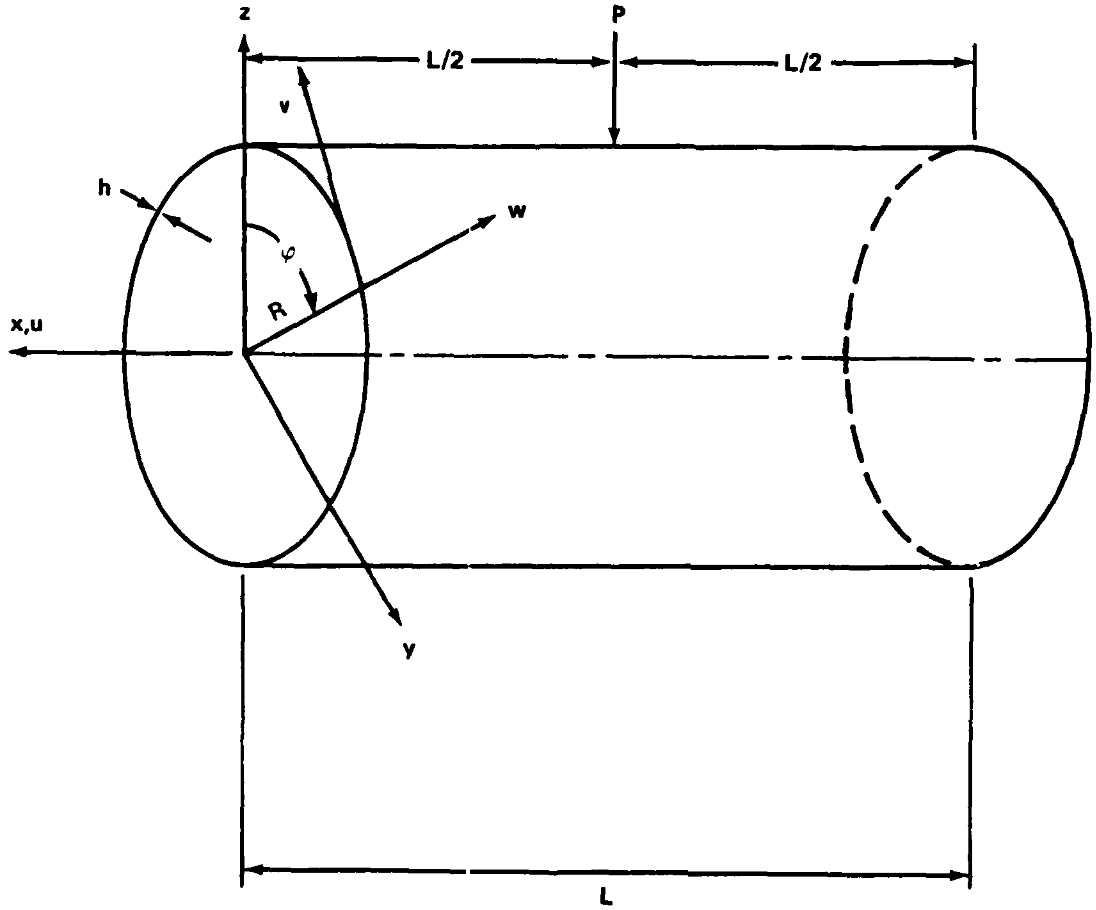

FIGURE B-1. CYLINDRICAL SHELL SIMPLY SUPPORTED AT BOTH ENDS AND LOADED BY A RADIALLY INWARDS CONCENTRATED FORCE P . SHELL IS OF RADIUS R, THICKNESS h, AND LENGTH L

where x is the longitudinal shell axis, φ the circumferential direction, and r the radial axis. The object of the following paragraphs is to eliminate the displacements u,v and obtain a partial differential equation for the radial deformation w, only involving the loading function p_r also. Because of the process of repeated differentiation, not all of the solutions of the reduced eighth order equation satisfy equations (B.1), (B.2), and (B.3). Additional solutions enter, which must be excluded by careful analysis.

The final partial differential equation, which was obtained after a lot of algebra and neglection of terms of order $[h/R]^4$ or higher, assumes the form

$$\nabla^8 w + \frac{12(1-\nu^2)}{R^2 h^2} \frac{\partial^4 w}{\partial x^4} + \frac{1}{R^2}\left[\frac{2}{R^6}\frac{\partial^6 w}{\partial \varphi^6} + \frac{(\nu+7)}{R^4}\frac{\partial^6 w}{\partial x^2 \partial \varphi^4} + \frac{(6+\nu-\nu^2)}{R^2}\frac{\partial^6 w}{\partial x^4 \partial \varphi^2}\right]$$

$$- \frac{1}{D}\nabla^4 p_r = 0 \qquad \text{(B.5)}$$

with

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Equation (B.5) was solved by the method of separation of variables through Fourier series (the fact that the P.D.E. contains even powers of w derivatives is decisive). In particular we may represent the radial load by a double Fourier series and similarly the radial displacement. Such an operation leads to a term by term equality of the coefficient of the series. Note, however, that, by resolving the load in such a form, boundary conditions and any symmetry or antisymmetry considerations must be satisfied. Unlike ordinary differential equations, where the homogeneous (complementary function) and particular solutions (particular integral) produce the solution, partial differential equations are substantially more difficult to solve and obtain general solutions. We are content to find just a solution.

Let the surface loading function p_r be resolved in a series of the form

$$p_r\ (x,\varphi) = \sum_m \sum_n Z_{m,n} \cos\ (m\varphi)\ \sin\left(\frac{\lambda x}{R}\right) \tag{B.6}$$

where $\lambda = \frac{n\pi R}{L}$ (B.7)

and

$$w(x,\varphi) = \sum_m \sum_n W_{m,n}\ \cos\ (m\varphi)\ \sin\left(\frac{\lambda x}{R}\right) \tag{B.8}$$

where $\sin[(\lambda/R)L] = \sin[n\pi] = 0$ (B.9)

This leads to

$$W_{m,n} = \frac{R^4\left(m^2 + \lambda^2\right)^2\ Z_{m,n}}{D\left\{\left(m^2+\lambda^2\right)^4 + 12(1 - \nu^2)\lambda^4 \left[\frac{R}{h}\right]^2 - m^2\left[2m^4 + (6+\nu-\nu^2)\lambda^4 + (7+\nu)m^2\lambda^2\right]\right\}} \tag{B.10}$$

Set $\alpha = L/R$ (B.11)

and $\gamma = R/h$ (B.12)

to obtain

$$W_{m,n} = \frac{\left(m^2\alpha^2 + n^2\pi^2\right)^2\ L^4}{DG}\ Z_{m,n} \tag{B.13}$$

where

$$G = (m^2\alpha^2 + n^2\pi^2)^4 + 12(1-\nu^2)n^4\pi^4\alpha^4\gamma^2 - m^2\alpha^4\left[2m^4a^4 + (6+\nu-\nu^2)n^4\pi^4\right.$$

$$\left. + (7+\nu)m^2\alpha^2n^2\pi^2\right] \tag{B.14}$$

To obtain the Fourier coefficients we may proceed as follows. By separation of variables resolve $p_r\ (x,s)$ into a product

$$p_r\ (x,s) = p(x)\ q(s) \tag{B.15}$$

where

$$q(s) = Q_0 \qquad \text{for } |s| \leq C_1$$
$$q(s) = 0 \qquad \text{otherwise} \tag{B.16}$$

and

$$p(x) = P_0 \qquad \text{for } b - \delta \leqq x \leqq b + \delta$$
$$p(x) = 0 \qquad \text{otherwise} \tag{B.17}$$

and $b = L/2$, where L is the entire length of the shell. Since $q(s)$ is even in the arc variable s, we may resolve it in a cosine Fourier series in the interval $(-\pi a, \pi a)$ with a period $T = 2\pi a$, i.e.,

$$q(s) = \frac{1}{2} q_o + \sum_{k=1,2,...}^{\infty} q_k \cos \frac{ks}{R} \tag{B.18}$$

where

$$q_0 = \frac{2}{L} \int_{-L/2}^{L/2} q(s)ds = \frac{4}{L} \int_{o}^{L/2} q(s)ds = \frac{2}{\pi R} Q_o C_1 \tag{B.19}$$

$$q_k = \frac{2}{L} \int_{-L/2}^{L/2} q(s) \cos\left(\frac{ks}{R}\right)ds = \frac{4}{L} \int_{o}^{1/2} q(s) \cos\left(\frac{ks}{R}\right)ds = \frac{2Q_o}{\pi k} \sin\left(\frac{kC_1}{R}\right) \tag{B.20}$$

Therefore

$$q(s) = \frac{2Q_o}{\pi R}\left\{\frac{C_1}{2} + \sum_{k=1,2,3}^{\infty} \frac{R}{k} \sin\left(\frac{kC_1}{R}\right) \cos\left(\frac{ks}{R}\right)\right\} \tag{B.21}$$

Similarly, $p(x)$ is resolved in an odd Fourier series in the interval $(-L,+L)$ with period $2L$ and $b=L/2$

$$p(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{B.22}$$

where

$$B_n = \frac{2}{L} \int_{o}^{L} q(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_{b-\delta}^{b+\delta} P_o \sin \frac{n\pi x}{L} dx$$

$$= \frac{4P_o}{n\pi} (-1)^{\frac{n-1}{2}} \quad \sin \frac{n\pi\delta}{L} \quad \text{for } n \text{ odd}$$

$$= 0 \qquad\qquad \text{for } n \text{ even} \tag{B.23}$$

Therefore

$$p(x) = \sum_{n=1,3,5}^{\infty} \frac{4P_o}{n\pi} (-1)^{\frac{n-1}{2}} \quad \sin \frac{n\pi\delta}{L} \sin \frac{n\pi x}{L} \tag{B.24}$$

Finally, the point load case can be arrived at by taking limts as both C_1 and δ tend to zero and

$$4Q_0P_0C_1\delta = P \tag{B.25}$$

i.e.,

$$p_r\ (x,s) = \frac{2P}{\pi RL} \left\{ \frac{1}{2} + \sum_{k=1,2,3,\ldots}^{\infty} \cos \left(\frac{ks}{R}\right) \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \right\} \sin \frac{n\pi x}{L} \tag{B.26}$$

or

$$p_r\ (x,\varphi) = \frac{2P}{\pi RL} \left\{ \frac{1}{2} + \sum_{k=1,2,3}^{\infty} \cos\ (k\varphi) \right\} \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \sin \frac{n\pi x}{L} \tag{B.27}$$

The Fourier coefficients $Z_{m,n}$ for the loading function of a radially concentrated load at the midpoint of span L (even series peripherally) are

$$Z_{m,n} = (-1)^{\frac{n-1}{2}} \frac{P}{\pi RL} \qquad \text{for } m=0 \qquad \text{and} \qquad n = 1,3,5,7,\ldots,$$

$$\tag{B.28}$$

$$Z_{m,n} = (-1)^{\frac{n-1}{2}} \frac{2P}{\pi RL} \qquad \text{for } m=1,2,3\ldots, \qquad \text{and} \qquad n = 1,3,5,\ldots,$$

Consequently, the radial deflection finally can be written as

$$w(x,\varphi) = \sum_{m=0,1}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{(m^2\alpha^2 + n^2\pi^2)^2}{DG} L^4 Z_{m,n} \cos(m\varphi) \sin\left(\frac{n\pi}{L} x\right) \tag{B.29}$$

MIZOGUCHI's SOLUTION

Mizoguchi[B-3] employed a slightly different form of the eighth order partial differential equation in radial displacement w (again even powers of w derivatives of w)

$$A \frac{\partial^8 w}{\partial x^8} + \frac{4B}{R^2} \frac{\partial^8 w}{\partial x^6 \partial \varphi^2} + \frac{6C}{R^4} \frac{\partial^8 w}{\partial x^4 \partial \varphi^4} + \frac{8-2\nu^2}{R^4} \frac{\partial^6 w}{\partial x^4 \partial \varphi^2} + \frac{6L}{R^4} \frac{\partial^4 w}{\partial x^4} + \frac{4}{R^6} \frac{\partial^2}{\partial x^2} \left[\frac{\partial^6 w}{\partial \varphi^6} + 2 \frac{\partial^4 w}{\partial \varphi^4} + \frac{\partial^2 w}{\partial \varphi^2} \right] + \frac{1}{R^8} \frac{\partial^2}{\partial \varphi^2} \left[\frac{\partial^6 w}{\partial \varphi^6} + 2 \frac{\partial^4 w}{\partial \varphi^4} + \frac{\partial^2 w}{\partial \varphi^2} \right] = \frac{1}{D} \left[A \frac{\partial^4 p}{\partial x^4} + \frac{2U}{R^2} \frac{\partial^4 p}{\partial x^2 \partial \varphi^2} + \frac{B}{R^4} \frac{\partial^4 p}{\partial \varphi^4} \right] \tag{B.30}$$

$$A = 1 + \frac{1}{3} \left[\frac{h}{R}\right]^2 , \quad B = 1 + \frac{1}{12} \left[\frac{h}{R}\right]^2 , \quad C = 1 + \frac{1-\nu^2}{72} \left[\frac{h}{R}\right]^2$$

$$L = 2(1-\nu^2) \left[\frac{R}{h}\right]^2 A , \qquad U = 1 + \frac{1 + (1-\nu)^2}{12(1-\nu)} \left[\frac{h}{R}\right]^2 \tag{B.31}$$

He applied a double Fourier transform to both sides of (B.30) of the form

$$\bar{w}(m,n) = \int_{-L}^{L} \int_{-\pi}^{\pi} w(x,\theta) \cos \frac{m\pi x}{L} \cos (n\varphi) \, d\varphi dx \tag{B.32}$$

He considered the cylindrical shell to be subjected to a series of point loads (Figure B-2), alternating from +P to -P every L. Consequently at $\pm L/2$

$$w = 0, \; v = 0, \; N_x = 0, \; M_x = 0 \tag{B.33}$$

At $\pm L$, where the deflection is maximum, the slope $\frac{\partial w}{\partial x}$ is zero. To use this method we need to know the values of the boundary terms $\frac{\partial^3 w}{\partial x^3}$ and $\frac{\partial^5 w}{\partial x^5}$ at $\pm L/2$.

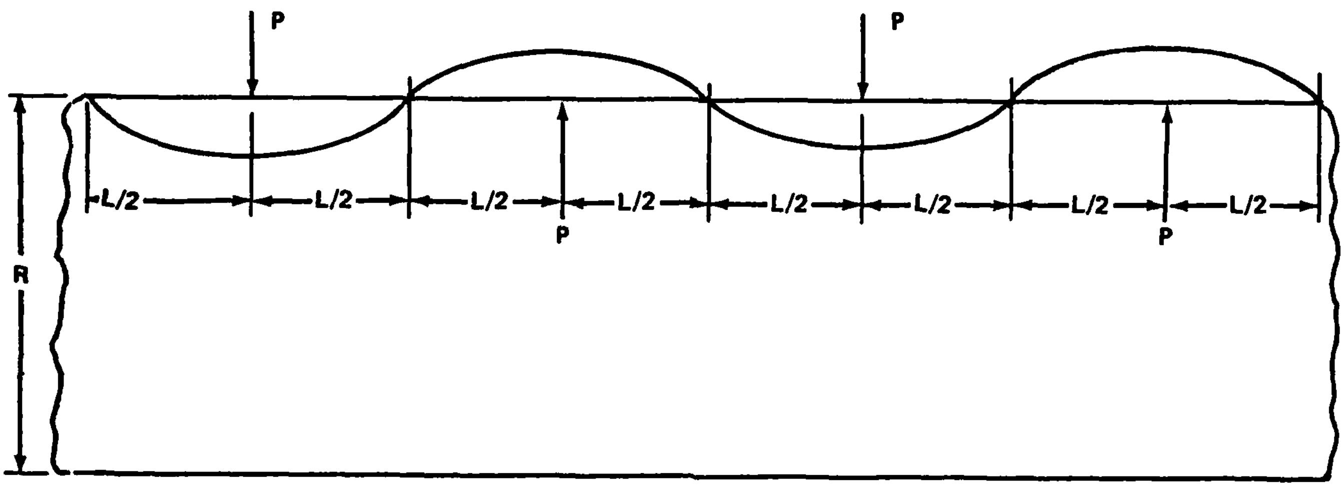

FIGURE B-2. SIMULATION OF SIMPLY SUPPORTED CONDITION BY CONSIDERING AN INFINITE NUMBER OF POINT LOADS P AT A DISTANCE L APART

On the other hand, if we proceed as Bijlaard,[B-1] i.e.,

$$p(x,\varphi) = \sum_n \sum_m Z_{m,n} \cos (n\varphi) \sin \frac{m\pi x}{L} \tag{B.34}$$

$$w(x,\varphi) = \sum_n \sum_m W_{m,n} \cos (n\varphi) \sin \frac{m\pi x}{L} \tag{B.35}$$

and set $\mu = m\pi R/L$ we get

$$W_{m,n} = \frac{R^4}{D} \frac{(A\mu^4 + 2U\mu^2 n^2 + Bn^4)}{H} Z_{m,n} \tag{B.36}$$

$$H = A\mu^8 + 4B\mu^6 n^2 + 6\, C\mu^4 n^4 - (8 - 2\nu^2)\, \mu^4 n^2 + 6L\mu^4 + 4\mu^2 (n^2-1)^2 n^2$$
$$+ n^4 (n^2 - 1)^2$$

and

$$Z_{m,n} = (-1)^{\frac{m-1}{2}} f \frac{P}{\pi RL} \quad \text{for} \quad \begin{array}{l} n = 0,1,2,\ldots \\ m = 1,3,5,7 \end{array} \tag{B.37}$$

with

$f = 1$ for $n = 0$
$f = 2$ for $n = 1,2,3...$

Therefore, the displacement under the point load P becomes, noting that, when the peripheral index n=0, the numerator and denominator simplifies

$$W\left(\frac{L}{2}, 0\right) = \frac{R^3 P}{\pi DL} \left\{ \sum_{m=1,3,5}^{\infty} \frac{1}{\left\{\mu^4 + 12(1-\nu^2)\left[\frac{R}{h}\right]^2\right\}} + 2 \sum_{m=1,3,5,}^{\infty} \sum_{n=1,2,3}^{\infty} \frac{(A\mu^4 + 2U\mu^2 n^2 + Bn^4)}{H} \right\} \tag{B.38}$$

where $(-1)^{\frac{n-1}{2}} \sin(m\pi/2)$ has been replaced by 1.0 for the odd values that m and n assume.

ODQVIST's SOLUTION

Odqvist[B-4] used Schorer's theory[B-5] to obtain the solution to a simply supported shell subject to a concentrated load P at midspan (also line load on generatrix). Starting from the simplified equation

$$\frac{\partial^8 M_\varphi}{\partial \varphi^8} + \frac{12(1-\nu^2)R^2}{h^2} \frac{\partial^4 M_\varphi}{\partial x^4} = 0 \tag{B.39}$$

and assuming a general term of the Fourier series to be

$$e^{A\varphi} \cos\left(\frac{n\pi x}{L}\right)$$

he determined in the next steps the form of the series. He used the principal roots (second quadrant as mentioned previously) from the auxiliary equation

$$A^8 + \left(\frac{n\pi}{L}\right)^4 \frac{12(1-\nu^2)R^6}{h^2} = 0 \tag{B.40}$$

$$\text{Setting } m^8 = 12(1-\nu^2)\left[\frac{R}{h}\right]^2\left[\frac{\pi R}{L}\right]^4 \tag{B.41}$$

we obtain

$$A^8 = -n^4 m^8 = e^{\pm J\pi}\, n^4 m^8 \tag{B.42}$$

or

$$A = \pm\, m\sqrt{n}\; e^{\pm J\frac{\pi}{8}} = \pm\, m\sqrt{n}\left(\cos\frac{\pi}{8} + J\sin\frac{\pi}{8}\right) \tag{B.43}$$

where $\dot{J} = \sqrt{-1}$

Choose

$$A = m\sqrt{n}\left(-\cos\frac{\pi}{8} + J\sin\frac{\pi}{8}\right) = \sqrt{n}\left(-m_1 + Jm_2\right) \tag{B.44}$$

Finally, the form of the Bending Moment/unit length M_φ is

$$M_\varphi(x,e) = \sum_{n=1,3,5}^{\infty} \left[e^{-m_1\sqrt{n}\ \varphi}\left(C_{1n} \sin\ (m_2\sqrt{n}\ \ \varphi) + C_{2n}\cos\left(m_2\sqrt{n}\ \ \varphi\right)\right)\right.$$

$$\left. + e^{-m_2\sqrt{n}\ \varphi}\left(D_{1n} \sin\ (m_1\sqrt{n}\ \ \varphi) + D_{2n} \cos\left(m_2\sqrt{n}\ \ \varphi\right)\right)\right] \cos\frac{n\pi x}{L} \tag{B.45}$$

Similar assumptions for the load and a lot of manipulation yields that the central radial displacement is

$$W(0,0) = \sum_{n=1,3,5}^{\infty} \frac{1}{n\sqrt{n}} \left\{ \frac{\sqrt{2-\sqrt{2}}\ \left[12(1-\nu^2)\right]^{5/8} R^{3/4} L^{1/2}\ P}{\sqrt{2\pi}\ \ 2\pi\ \ h^{9/4}\ \ \ \ E} \right\} \tag{B.46}$$

Because the computation of this infinite series is extremely slow (over 20,000 terms) we may proceed as follows to evaluate the series.

From Whittaker & Watson[B-6] we have that the definition of the Zeta function is

$$\zeta(s) = \sum_{n=1,2,\ldots}^{\infty} 1/n^s \tag{B.47}$$

where s could be complex. This is an analytic, uniformly convergent series for $Re(s) \geq 1+\delta$. Furthermore, by the Jensen formula,[B-6] (p. 279)

$$\zeta(s) = \frac{2^{s-1}}{(s-1)} - 2^s \int_0^\infty \frac{\sin\ [s\ \arctan y]}{[1+y^2]^{s/2}\ [1+e^{\pi y}]}\ dy$$

$$= \frac{2^{s-1}}{(s-1)}\left[1 - \frac{(s-1)}{2^{s-1}}\ 2^s \int_0^\infty \frac{\sin[s\arctan y]}{[1+y^2]^{s/2}[1+e^{\pi y}]}\ dy\right]$$

$$= \frac{2^{s-1}}{(s-1)}\left[1 - 2(s-1)\int_{\theta=0}^{\theta=\pi/2} \frac{\sin\ \ (s\theta)\ \ \ \ d\theta}{[1+\tan^2\theta]^{s/2}\ [1+e^{\pi\tan\theta}]\cos^2\theta}\right] \tag{B.48}$$

We note that as $\theta \to 0$, the integrand $\to s\theta/2$, while the portion of the integrand

$$[1 + \exp(\pi \tan \theta)] \cos^2\theta \to \frac{\pi^2}{2}$$

However,

$$1 - \frac{1}{2^s} \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \sum_{k=o}^{\infty} \frac{1}{(2k+1)^s} \tag{B.49}$$

$$\text{Therefore} \quad [1 - 2^{-s}] \zeta(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} \tag{B.50}$$

for $Re(s) \geq 1 + \delta$, where δ is positive. We may calculate $\zeta(3/2)$ and thus obtain the sum to our series. The integral of (B.48) is performed numerically by the trapezoidal rule.

OTHER METHODS

The following brief account will be outlined with respect to some of the above mentioned solution procedures.

Initially Yuan[B-7] employed the simplified Donnell equation of the form

$$\nabla^8 w + \frac{12(1-\nu^2)}{R^2 h^2} \frac{\partial^4 w}{\partial x^4} - \frac{1}{D} \nabla^4 q = 0 \tag{B.51}$$

to obtain a solution of the infinitely long cylinder subject to two equal and opposite point loads. The loading function q(x,s) was resolved in a product of a Fourier integral axially and a cosine Fourier series peripherally. The loading function q(x,s) in radial load is even in both x and s variables. In their follow-up paper Ting & Yuan[B-8] used the complete Donnell equation, which can be written as follows

$$\nabla^8 w + \frac{12(1-\nu^2)}{R^2 h^2} \frac{\partial^4 w}{\partial x^4} + \frac{2}{R^8} \frac{\partial^6 w}{\partial \varphi^6} + \frac{1}{R^8} \frac{\partial^4 w}{\partial \varphi^4} - \frac{1}{D} \nabla^4 q = 0 \tag{B.52}$$

They dealt with boundary conditions other than simply supported. They employed the infinite length cylinder solution as a particular integral. Boundary conditions at the ends were satisfied by obtaining proper solutions to the homogeneous differential equation.

Finally Yuan & Ting[B-9] employed the more accurate Fluegge equation

$$\nabla^8 w + \frac{12(1-\nu^2)}{R^2h^2}\frac{\partial^4 w}{\partial x^4} + \frac{2}{R^2}\frac{\partial^6 w}{\partial s^6} + 2(4-\nu)\frac{\partial^6 w}{\partial s^4 \partial x^2} + 6\frac{\partial^6 w}{\partial s^2 \partial x^4} + 2\nu\frac{\partial^6 w}{\partial x^6}$$

$$+ \frac{1}{R^4}\frac{\partial^4 w}{\partial s^4} + 2\,(2-\nu)\frac{\partial^4 w}{\partial s^2 \partial x^2} - \frac{1}{D}\nabla^4 q = 0 \tag{B.53}$$

The method of images was employed to obtain the radial displacement of a simply supported cylinder under the applied load as

$$\frac{wEh}{P} = \frac{6(1-\nu^2)}{\pi}\left[\frac{R}{h}\right]^2 \sum_{n=2,4,6}^{\infty}\left[\left\{\frac{\cosh\left(B_1\frac{L}{R}\right) - \cos\left(A_1\frac{L}{R}\right)}{\sinh^2\left(B_1\frac{L}{R}\right) + \sin^2\left(A_1\frac{L}{R}\right)}\right\}\right.$$

$$\left\{M_1\sinh\left(B_1\frac{L}{R}\right) - M_2\sin\left(A_1\frac{L}{R}\right)\right\} + \left\{\frac{\cosh\left(G_1\frac{L}{R}\right) - \cos\left(C_1\frac{L}{R}\right)}{\sinh^2\left(G_1\frac{L}{R}\right) + \sin^2\left(C_1\frac{L}{R}\right)}\right\}$$

$$\left.\left\{M_3\sinh\left(G_1\frac{L}{R}\right) - M_4\sin\left(C_1\frac{L}{R}\right)\right\}\right] \tag{B.54}$$

where all parameters are defined in Reference B-9 or in main text.

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APPENDIX C

SYMMETRY OF VLASOV SHELL THEORY

The following statements are made to establish an important point as far as the form of the used shell equations is concerned. It concerns the symmetry of the coefficient matrix.

Written in matrix notation, but with their respective sign convention maintained, the thin shell Timoshenko equations[C-1] used in Biljaard's solution,[C-2] for radial load p_r acquire the form used in Equation (C.1). Notice the asymmetry of the coefficient matrix in terms $C_{3,2}$ and $C_{2,3}$. Fluegge's equations[C-3] for the same problem in his notation and sign convention are shown in Equations (C.2) and (C.3). The coefficient matrix is fully symmetrical.

The same is true with Vlasov equations.[C-4,C-5] They are symmetrical and can be displayed as shown in Equation (C.4). The important point, which more often than not is missed, is that the reciprocity law of Betti does not hold for Timoshenko equations.[C-1] Within the limits of shell theory, however, it does not restrict their application in the "engineering" sense. Finally, it should be clear that Fluegge,[C-3] Vlasov,[C-4,C-5] or other types of shell equations ought to be preferred due to symmetry considerations.

$$\begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{(1-\nu)}{2a^2}\frac{\partial^2}{\partial \varphi^2} & \frac{(1+\nu)}{2a}\frac{\partial^2}{\partial x \partial \varphi} & -\frac{\nu}{a}\frac{\partial}{\partial x} \\ \frac{(1+\nu)}{2a}\frac{\partial^2}{\partial x \partial \varphi} & \frac{(1-\nu)}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left[(1-\nu)\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2}\right] & -\frac{1}{a^2}\frac{\partial}{\partial \varphi} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left[\frac{\partial^3}{\partial x^2 \partial \varphi} + \frac{1}{a^2}\frac{\partial^3}{\partial \varphi^3}\right] \\ -\frac{\nu}{a}\frac{\partial}{\partial x} & -\frac{1}{a^2}\frac{\partial}{\partial \varphi} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left[(2-\nu)\frac{\partial^3}{\partial x^2 \partial \varphi} + \frac{1}{a^2}\frac{\partial^3}{\partial \varphi^3}\right] & \frac{1}{a^2} + \frac{1}{12}h^2\nabla^4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} o \\ o \\ \frac{(1-\nu^2)P_r}{Eh} \end{bmatrix} \tag{C.1}$$

$$\begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{(1-\nu)}{2a^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{12}\left(\frac{h}{a}\right)^2\frac{(1-\nu)}{2a^2}\frac{\partial^2}{\partial \varphi^2} & \frac{(1+\nu)}{2a}\frac{\partial^2}{\partial x \partial \varphi} & \frac{\nu}{a}\frac{\partial}{\partial x} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left[-a\frac{\partial^3}{\partial x^3} + \frac{(1-\nu)}{2a}\frac{\partial^3}{\partial x \partial \varphi^2}\right] \\ \frac{(1+\nu)}{2a}\frac{\partial^2}{\partial x \partial \varphi} & \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2} + \frac{(1-\nu)}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{12}\left(\frac{h}{a}\right)^2\frac{3}{2}(1-\nu)\frac{\partial^2}{\partial x^2} & \frac{1}{a^2}\frac{\partial}{\partial \varphi} - \frac{1}{12}\left(\frac{h}{a}\right)^2\frac{(3-\nu)}{2}\frac{\partial^3}{\partial x^2 \partial \varphi} \\ \frac{\nu}{a}\frac{\partial}{\partial x} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left[\frac{(1-\nu)}{2a}\frac{\partial^3}{\partial x \partial \varphi^2} - a\frac{\partial^3}{\partial x^3}\right] & \frac{1}{a^2}\frac{\partial}{\partial \varphi} - \frac{1}{12}\left(\frac{h}{a}\right)^2\frac{(3-\nu)}{2}\frac{\partial^3}{\partial x^2 \partial \varphi} & \frac{1}{a^2} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left[a^2\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial \varphi^2} + \frac{1}{a^2}\frac{\partial^4}{\partial \varphi^4} + \frac{2}{a^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{a^2}\right] \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{P_x}{D} \\ -\frac{P_\varphi}{D} \\ \frac{P_r}{D} \end{bmatrix} \tag{C.2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2} \tag{C.3}$$

$$\begin{bmatrix} \frac{\partial^2}{\partial\alpha^2} + \frac{(1-\nu)}{2}\frac{\partial^2}{\partial\beta^2} & \frac{(1+\nu)}{2}\frac{\partial^2}{\partial\alpha\partial\beta} & \nu\frac{\partial}{\partial\alpha} - \frac{1}{12}\left(\frac{h}{a}\right)^2\left[\frac{\partial^3}{\partial\alpha^3} - \frac{(1-\nu)}{2}\frac{\partial^3}{\partial\alpha\partial\beta^2}\right] \\ \frac{(1+\nu)}{2}\frac{\partial^2}{\partial\alpha\partial\beta} & \frac{\partial^2}{\partial\beta^2} + \frac{(1-\nu)}{2}\frac{\partial^2}{\partial\alpha^2} & \frac{\partial}{\partial\beta} - \frac{(3-\nu)}{2}\frac{1}{12}\left(\frac{h}{a}\right)^2\frac{\partial^3}{\partial\alpha^2\partial\beta} \\ \nu\frac{\partial}{\partial\alpha} - \frac{1}{12}\left(\frac{h}{a}\right)^2\left[\frac{\partial^3}{\partial\alpha^3} - \frac{(1-\nu)}{2}\frac{\partial^3}{\partial\alpha\partial\beta^2}\right] & \frac{\partial}{\partial\beta} - \frac{(3-\nu)}{2}\frac{1}{12}\left(\frac{h}{a}\right)^2\frac{\partial^3}{\partial\alpha^2\partial\beta} & \frac{1}{12}\left(\frac{h}{a}\right)^2\left[\nabla^4 + 2\frac{\partial^2}{\partial\beta^2} + 1\right] + 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{(1-\nu^2)}{Eh}a^2X \\ \frac{(1-\nu^2)}{Eh}a^2Y \\ -\frac{(1-\nu^2)}{Eh}a^2Z \end{bmatrix} \tag{C.4}$$

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